INTERNATIONAL JOURNAL OF NOVEL RESEARCH AND DEVELOPMENT (IJNRD)
© 2026 IJNRD | Volume 11, Issue 5, May 2026 | ISSN: 2456-4184 | IJNRD.ORG

A Survey on Asymptotic and Exponential Stability of Nonlinear Differential Equations

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1. Abstract

The investigation of asymptotic and exponential stability, one of the most basic problems existing in a qualitative theory for nonlinear differential equations, The proliferation of dynamical systems of increasing sophistication in engineering, biology, physics, and control science has created an enormous need for stability frameworks that are not only rigorous and systematic but also computationally tractable. This survey gives a detailed, critical review of theoretical results, recent developments and applications for asymptotic and exponential stability of nonlinear ordinary and partial differential equations. Starting with a rigorous approach to the mathematical definition of equilibrium definitions, Lyapunov stability and classical hierarchy of stability concepts, the survey methodically covers asymptotic stability via Lyapunov's direct method, LaSalle's invariance principle and perturbation in Section 5, before addressing the topic of exponential stability through decay rate bounds (uniform versus non uniform), as well as Lyapunov based sufficient conditions. The analytical methods are assessed, along with some of their strengths and well established limitations, by regarding approaches that include linearization, Lyapunov function construction algorithms, fixed point theorems and comparison principles. In terms of high dimensional and non smooth systems, numerical and computational approaches, such as simulation based verification (SBV) and software assisted analysis, are reviewed. Domain specific examples are handled spanning application domains including control engineering, power systems, robotics, biological modeling and mechanical systems. They close the work with an organized synthesis of recent developments, highlighting open challenges and proposing frontier research areas including (but not limited to) stability of stochastic systems, high dimensional dynamic systems, and novel structured Lyapunov function synthesis methods based on large amounts of data. The work is to be useful as a reference for researchers working in this space (and also to be analytically rigorous enough to serve as an entry point into the field).

Keywords: Asymptotic Stability; Exponential Stability; Lyapunov Methods; Nonlinear Differential Equations; LaSalle's Invariance Principle; Lyapunov Function Construction; Stability of Nonlinear Systems; Control Systems; Perturbation Theory; Stochastic Stability

2. Introduction

2.1 Background and Motivation

The analysis of stability in dynamical systems has intellectual roots in the foundational work of Aleksandr Mikhailovich Lyapunov, whose 1892 doctoral dissertation laid much of the theoretical groundwork that remains relevant for modern stability theory today. The key insight of Lyapunov that scalar energy like functions could characterize the behavior of trajectories near an equilibrium point without having to explicitly know formulae for the solutions marked a major shift away from the traditional paradigm of explicit integration. Stability theory has, in the century since that original contribution, become a highly sophisticated field incorporating concepts from differential equations, functional analysis, control theory and computational mathematics.[1]

In Section II, we discuss the motivation behind continuous research in the asymptotic and exponential stability of nonlinear differential equations from theoretical imperatives as well as urgent practical necessities. At a theoretical level, the nonlinear regime has challenges of a different qualitative nature than the linear case. In contrast to linear systems, where we have complete spectral characterizations of stability in terms of eigenvalues, it is well known that nonlinear systems may exhibit behaviors (e.g. bifurcations, limit cycles), chaos and other types of attractors as well as finite time blow up and multistability; direct linear methods 3 can be insufficient or unreliable for these types of simple but significant dynamics. The continued shortage of a universal nonlinear system stability criteria has inspired decades of methodological development, ranging from the classical Lyapunov perspective to contemporary sum of squares programming and machine learning assisted Lyapunov synthesis.[2]

From an application point of view, the result of such instability translates to catastrophic results in real systems. The cascading failure of an interconnected power network during the 2003 Northeast North American blackout illustrates the destructive potential of instability in some nonlinear dynamical systems. The attitude control system achieving or failing a mission is governed by stability in aerospace engineering; the organs/parts actually working as intended or not, e.g. a drug overdosing and losing potency in place of effectiveness and leading to side effects, comes down to dynamics versus threshold level (stable); or an ecosystem being stable and persistent rather than vulnerable depending on demographic parameters all this is determined through stability analysis. Taken together, these applications require stability frameworks that are not just qualitatively informative but quantitatively accurate able to deliver decay rates, robustness margins and performance guarantees.

In this respect, the difference between asymptotic stability and exponential stability is quite significant. Asymptotic stability is the property that the system returns to its equilibrium because as time $\to \infty$, it converges (which means that at $t = \infty$ all the trajectories are centered in equilibrium) but there is no constraint on how fast does that happen. In contrast to mere stability, exponential stability guarantees that the object converges uniformly at a geometric rate (the basis we require for engineering design specifications). Hence determining when stabilization can be achieved exponentially or not is not only an academic interest but something that has direct engineering relevance.[3]

Over the past years, we have seen remarkable methodological expansion. The advent of sum of squares (SOS) optimization has made it possible to compute polynomial Lyapunov functions. Then, the newly developed method for Lyapunov function synthesis based on neural networks has opened up stability analysis of high dimensional dynamics from data. Refined Converse Lyapunov theorems more sharply characterize when a

Lyapunov function is guaranteed to exist.[4] The stability theory of fractional order, time delay and hybrid systems has been more or less mature. Together, these advances characterize a research space that is both vibrant and technically challenging, as well as practically consequential.

2.2 Importance of Stability Analysis

Stability analysis is a next to the brazenly trivial use of mathematical sciences, but this has roots more profound than whether a trajectory comes back to an equilibrium or not.[5] Stability analysis, in its most basic form, is concerned with the stability of mathematical models asking if a differential equation (and thus the actual physical system it represents) will possess dynamics that are reasonably predictable and physically interpretable under realistic conditions of initialization and perturbation.[5]

When it comes to control systems engineering, stability is a prerequisite of any substantial design effort. Irrespective of the input performance characteristics, a feedback controller that cannot ensure closed loop system stability is essentially unacceptable.[6] This separation of stabilization from performance optimization, which is the basis for much of both classical and modern control theory, puts stability analysis as the mathematically less important problem. In parallel with this, the nonlinear control paradigms such as feedback linearization, backstepping, sliding mode control and model predictive control emerged that call for guarantees of stability[7]. The results from the academic literature in terms of stability analysis tools thus provide direct pipeline links into industrial control design[8].

Stable equilibrium in biological and ecological modeling is the conceptual overhead of stability analysis, which addresses resilience, persistence and vulnerability. Coexistence can be a robust or fragile outcome depending on whether the equilibrium in a predator prey model is stable. A public health problem of manifest significance is whether an infection will die out or persist, a question determined by the stability of the disease free equilibrium in epidemiological models.[9] AbstractThe stability analysis of compartmental epidemic models and, in particular, SEIR and SEIRS models under different control interventions attracted renewed attention during the COVID 19 pandemic[10].

Stability analysis came into play to decide if the vibrations would be damped or amplified, whether a structural deformation will recover or escalate, and whether rotating machinery will continue working stably upon load fluctuations. The stability of power electronics and power systems in electrical engineering is a question of grid reliability. Nonlinear dynamics of synchronous generators, inverters, and transmission networks are currently active areas of stability research with direct relevance to energy infrastructure.[11]

The growing significance of stability analysis is accentuated by the emergence of networked, distributed and cyber physical systems where local stability properties have to be composed (or aggregated) in order to provide global guarantees. Of particular relevance in this context, input to state stability (ISS) and its variants provide stability guarantees with respect to external inputs important for systems interconnection and robustness analysis.[12]

2 Methods: Review Scope and Objectives

This survey focuses on the asymptotic and exponential stability of nonlinear differential systems, which include not only ordinary differential equations (ODEs) but, when appropriate, also partial differential equations (PDEs).[13] The members of the review team have done an outstanding job organizing the review to serve multiple constituencies researchers looking for a current synthesis of methodological advances, practitioners needing stable foundations for applied analysis, and graduate students who need a pedagogically organized entry into the field.[14]

Many of these limits are demarcated for the review in important ways. The first is that, although stability theory applies to difference equations, hybrid systems and delay differential equations, these topics are only covered in as far as is necessary to set the context or provide an appropriate comparison they represent secondary importance. Second, it focuses on finite dimensional and infinite dimensional continuous time systems are the primary focus; some parts of discrete time stability theory are mentioned tangentially. The intent here is not to examines stochastic stability as a main technical theme, but rather as part of open problems and future work emerging themes, while ensuring analytical focus.[15]

The objectives of this review are as follows:

(i) To give now the most comprehensive presentation of the mathematical foundations asymptotic and exponential stability for nonlinear differential equations all fully self contained, accessible to anyone with graduate training in mathematics or engineering.

(ii) "To conduct a critical survey of the most important analytical tools including Lyapunov's direct method, LaSalle's invariance principle, linearization, comparison principles and fixed point methods" with his specialties.

(iii) To provide a brief overview of new contributions (2020–2026) for stability analysis of nonlinear systems, showing the main methodological trends and problems that shape modern research.

Survey of computational and numerical methods for stability verification, with a critical discussion on the limits of these techniques when applied over high dimensional and/or non smooth systems.

(v) If to have a record of the usage of stability theory in engineering domains, biological and physical.

(vi) To discover open problems, research gaps and the most suitable directions for future work.

This review does not attempt to be comprehensive with respect to the literature, particularly because the literature on nonlinear stability is far too large for that goal to be a reasonable one. Rather, it intends analytically synthetic: identifying a conceptual structure for the field, placing each contribution within that structure and providing a critical assessment of the contributions in that light as an incumbent researcher working on nonlinear dynamical systems.[16]

3. Preliminaries and Mathematical Foundations

3.1 Basic Definitions and Notations

We define the harmonic notation and key baseline definitions that will be used for the rest of this review in Section [■]. Overall, this section can be regarded as a reference for the developments to follow, and also formulated very distinctly basic objects of study.

Notation: Denote by $\mathbb{R}$ as the real numbers, $\mathbb{R}^n$ as n dimensional Euclidean space, and $\mathbb{R}{+} = [0,\infty)$ as the nonnegative real line. For $x \in \mathbb{R}^n$, we write $|x|$ for the Euclidean norm, but all results stated in terms of norms are valid using any equivalent norm in finite dimensions. $|A|$: induced matrix norm for $A \in \mathbb{R}^{n \times n}$; $\lambda{\min}(A)$, $\lambda_{\max}(A)$: minimum and maximum eigenvalue of A, respectively when A is symmetric.

The open ball of radius $r > 0$ based at a point $x_0 \in \mathbb{R}^n$ is

B(x0,r)={x∈Rn:|x−x0|<r}.\mathcal{B}(x_0,r)=\{x\in\mathbb{R}^n:|x x_0|<r\}.B(x0,r)={x∈Rn:|x−x0|<r}.

Definition 3.1 (Comparison Functions). A function

α:R+→R+\alpha:\mathbb{R}_{+}\to\mathbb{R}_{+}α:R+→R+

belongs to class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0)=0$. It belongs to class $\mathcal{K}_{\infty}$ if additionally

α(s)→∞ass→∞.\alpha(s)\to\infty \quad \text{as} \quad s\to\infty.α(s)→∞ass→∞.

A function

β:R+×R+→R+\beta:\mathbb{R}_{+}\times\mathbb{R}_{+}\to\mathbb{R}_{+}β:R+×R+→R+

belongs to class $\mathcal{KL}$ if $\beta(\cdot,t)\in\mathcal{K}$ for each fixed $t\ge0$, and for each fixed $s>0$, the second argument of the above function when keeping constant is strictly decreasing to zero.

Class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ comparison functions form an essential toolbox of contemporary nonlinear stability analysis. This way, stability definitions can be formulated in a form that is independent of particular norms and can be generalized to infinite dimensional settings.

Function $V : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set, is called positive definite on $\mathcal{D}$ about a point $x_0 \in \mathcal{D}$, if $V(x_0)=0$ and $V(x)>0$, in $\mathcal{D}\setminus{x_0}$. If $ V$ is positive definite, then it is negative semidefinite if

V(x)≥0V(x)\ge0V(x)≥0

for all the new state space $\mathcal{D}$.

For this review, we will assume dynamical systems of the form of the autonomous ordinary differential equation unless specified otherwise:

x˙(t)=f(x(t)),x(0)=x0,(3.1)\dot{x}(t)=f(x(t)), \qquad x(0)=x_0, \tag{3.1}x˙(t)=f(x(t)),x(0)=x0,(3.1) where $x:\mathbb{R}^{+}\to\mathbb{R}^n$ is the state trajectory, $f:\mathcal{D}\to\mathbb{R}^n$ is a locally Lipschitz continuous vector field on an open set $\mathcal{D}\subseteq\mathbb{R}^n$ and $x_0\in\mathcal D$ is the initial condition. The Picard–Lindelöf theorem guarantees existence and uniqueness of solutions on a maximal interval of existence $[0,T_{\max})$, in which the local Lipschitz condition on f holds.[17]

For non autonomous systems we write:

x˙(t)=f(t,x(t)),x(t0)=x0,(3.2)\dot{x}(t)=f(t,x(t)), \qquad x(t_0)=x_0, \tag{3.2}x˙(t)=f(t,x(t)),x(t0)=x0,(3.2)

In fact, it can be an abstract situation let

f:R+×D→Rnf:\mathbb{R}_{+}\times\mathcal{D}\to\mathbb{R}^{n}f:R+×D→Rn

with some appropriate measurability and Carathéodory condition on f so that the existence of solutions is guaranteed.

Let $\phi(t,x_0)$ be the flow map or transition map of the system (3.1), which denotes the unique solution at time t at time 0, starting from x_0. For systems which are not autonomous, we denote by $\phi(t,t_0,x_0)$ the state of system at time t, initialized at time t_0 and initial condition x_0.[18]

3.2 Types of Differential Equations

The stability theory reviewed here is applicable to differential equations of many different but related structural types, and a better taxonomy would help position the subsequent technical advances.[19]

Ordinary Differential Equations (ODEs)

The ODEs are finite dimensional since (3.1) and (3.2) are systems of ODEs. System (3.1) is autonomous the vector field f does not depend on time explicitly, whereas (3.2) is nonautonomous. For autonomous systems, equilibrium points, limit cycles and other invariant sets serve as important organizing structures while even in the absence of external inputs non autonomous systems can behave in a time varying manner, leading to additional considerations in defining and verifying stability.[20]

One subclass of ODEs which is of particular importance is that of perturbed systems:

x˙(t)=f(x(t))+g(t,x(t)),(3.3)\dot{x}(t)=f(x(t))+g(t,x(t)), \tag{3.3}x˙(t)=f(x(t))+g(t,x(t)),(3.3)

where f represents the nominal dynamics and there is a perturbation term g, potentially unknown but bounded. Another subtopic in nonlinear stability theory is the question of how “stable” an equilibrium is under perturbations, defining notions such as input to state stability and ultimate boundedness.

Time delay differential equations (DDEs) can be described as follows:

x˙(t)=f(x(t),x(t−τ)),(3.4)\dot{x}(t)=f(x(t),x(t \tau)), \tag{3.4}x˙(t)=f(x(t),x(t−τ)),(3.4) where $\tau>0$ is a constant delay form a notable subclass which arises in communication networks, biological systems and process control. Lyapunov functions cannot suffice; they must be Lyapunov–Krasovskii functionals for DDEs and SDRFs. Although a full exposition of DDEs is not the main focus of this review, results linking DDE stability to finite dimensional settings will be included where appropriate.[21]

Partial Differential Equations (PDEs)

If the state varies both in a spatial domain $\Omega \subseteq \mathbb{R}^d$ as well as in time, then we have a PDE system. Consider a general evolution equation on a Banach or Hilbert space $\mathcal{H}$, of the form:

∂u∂t=Au+F(u),u(·,0)=u0,(3.5)\frac{\partial u}{\partial t}=\mathcal{A}u+\mathcal{F}(u), \quad u(\cdot,0)=u_0, \tag{3.5}∂t∂u=Au+F(u),u(·,0)=u0,(3.5)

where $\mathcal{A}$ is a (possibly non bounded) linear operator, while $\mathcal{F}$ is a nonlinear operator. In this sense, a stability analysis for PDEs typically involves the usage of functional analytic tools such as semigroup theory and Lyapunov functionals on an infinite dimensional space. The stability of partial differential equations (PDEs) arises in many applications such as thermoelasticity, fluid dynamics and reaction diffusion equations in population biology.[15]

Fractional Order Differential Equations

A extensive body of literature exists pertaining to the stability of fractional order systems:

CDtαx(t)=f(x(t)),0<α<1.(3.6){}^C D_t^\alpha x(t)=f(x(t)), \qquad 0<\alpha<1. \tag{3.6}CDtα x(t)=f(x(t)),0<α<1.(3.6)

where ${}^C D_t^\alpha$ denotes the Caputo fractional derivative. In the past decade, the theory of stability for such systems has progressed significantly with Lyapunov methods modified for the fractional setting.

Stochastic Differential Equations (SDEs)

When uncertainty or noise is modeled explicitly, systems assume the form of Itô stochastic differential equations:

dx(t)=f(x(t)) dt+σ(x(t)) dW(t),(3.7)dx(t)=f(x(t))\,dt+\sigma(x(t))\,dW(t), \tag{3.7}dx(t)=f(x(t))dt+σ(x(t))dW(t),(3.7)

where $W(t)$ is a standard Brownian motion and $\sigma:\mathbb{R}^n\to\mathbb{R}^{n\times m}$ is the diffusion matrix. Stability in probability, almost sure stability, and mean square stability are different notions in this context and their treatment requires stochastic Lyapunov methods.[22]

3.3 Concepts of Stability

3.3.1 Equilibrium Points

The analysis of stability for a differential equation necessarily begins with the identification of equilibrium points the constant solutions around which stability behavior is assessed.

Definition 3.2 (Equilibrium Point). A point $x^*\in\mathcal{D}$ is an equilibrium point (or rest point, singular point) of system (3.1) if

f(x∗)=0,f(x^*)=0,f(x∗)=0,

so that the constant function

x(t)≡x∗x(t)\equiv x^*x(t)≡x∗

is a solution.

Without loss of generality, by the coordinate transformation

y=x−x∗,y=x x^*,y=x−x∗,

we may assume that the equilibrium of interest is at the origin:

x∗=0,f(0)=0.x^*=0, \qquad f(0)=0.x∗=0,f(0)=0.

3.3.2 Lyapunov Stability

Lyapunov stability formalizes the idea that small perturbations from equilibrium should generate only small deviations in trajectories.

Definition 3.3 (Lyapunov Stability). The equilibrium $x^*=0$ of system (3.1) is Lyapunov stable if for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that:

|x(0)|<δ ⟹ |x(t)|<ε,∀t≥0.(3.8)|x(0)|<\delta \implies |x(t)|<\varepsilon, \qquad \forall t\ge0. \tag{3.8}|x(0)|<δ⟹|x(t)|<ε,∀t≥0.(3.8)

The equilibrium is unstable if it is not Lyapunov stable.

Lyapunov stability is qualitative and non asymptotic: it guarantees trajectories remain near equilibrium but does not imply convergence.

Definition 3.4 (Uniform Lyapunov Stability). The equilibrium $x^*=0$ of the non autonomous system (3.2) is uniformly stable if the δ in Definition 3.3 can be selected independently of t_0.[15]

3.3.3 Asymptotic Stability

Asymptotic stability extends Lyapunov stability by additionally requiring convergence to equilibrium.

Definition 3.5 (Local Asymptotic Stability). The equilibrium $x^*=0$ of system (3.1) is locally asymptotically stable (LAS) if it is Lyapunov stable and there exists $\delta_0>0$ such that:

|x(0)|<δ0 ⟹ lim⁡t→∞x(t)=0.(3.9)|x(0)|<\delta_0 \implies \lim_{t\to\infty}x(t)=0. \tag{3.9}|x(0)|<δ0 ⟹t→∞limx(t)=0.(3.9)

Definition 3.6 (Global Asymptotic Stability). The equilibrium $x^*=0$ is globally asymptotically stable (GAS) if it is Lyapunov stable and

lim⁡t→∞x(t)=0\lim_{t\to\infty}x(t)=0t→∞limx(t)=0

for all $x(0)\in\mathbb{R}^n$.

Using comparison functions, LAS may equivalently be characterized by the existence of a class $\mathcal{KL}$ function β and $\delta_0>0$ such that:

|x(0)|<δ0 ⟹ |x(t)|≤β(|x(0)|,t),∀t≥0.(3.10)|x(0)|<\delta_0 \implies |x(t)|\le\beta(|x(0)|,t), \qquad \forall t\ge0. \tag{3.10}|x(0)|<δ0⟹|x(t)|≤β(|x(0)|,t),∀t≥0.(3.10)

This characterization, although equivalent to the ε δ and convergence definition, is better suited to analysis and synthesis.[23]

3.3.4 Exponential Stability

Exponential stability offers the limitation for rate of convergence in the form of a quantitative requirement, namely that the decay be dominated by a negative exponential.

Definition 3.7 (Local Exponential Stability). System (3.1) has a locally exponentially stable equilibrium $x^*=0$ if there exist constants $c,\lambda,\delta_0>0$, such that:

|x(0)|<δ0 ⟹ |x(t)|≤c |x(0)| e−λt,∀t≥0.(3.11)|x(0)|<\delta_0 \implies |x(t)|\le c\,|x(0)|\,e^{ \lambda t}, \qquad \forall t\ge0. \tag{3.11}|x(0)|<δ0⟹|x(t)|≤c|x(0)|e−λt,∀t≥0.(3.11)

The positive constant λ is the decay rate (or convergence rate), and $c\ge1$ is an overshoot constant. If for all $x(0)\in\mathbb{R}^n$, (3.11) holds, the system is globally exponentially stable (GES).[15]

Exponential stability is strictly stronger than asymptotic stability: every exponentially stable equilibrium is asymptotically stable, but the converse fails[24]. An example given in classical texts is the function

x˙=−x3,\dot{x}= x^3,x˙=−x3,

whose equilibrium point at the origin is globally asymptotically stable, with solution

x(t)=(x0−2+2t)−1/2,x(t)=(x_0^{ 2}+2t)^{ 1/2},x(t)=(x0−2+2t)−1/2,

which decays algebraically rather than exponentially, hence not exponentially stable[25].

The applications place great importance on exponential stability. Using the exponential decay bound (3.11) we have:

1. Robustness under small perturbations: Exponentially stable systems remain asymptotically stable under sufficiently small Lipschitz perturbation[26].
2. Input to state stability: An unforced system is exponentially stable if and only if it can cope globally with additive disturbances (ISS), provided its perturbation structure satisfies minimal conditions[27].
3. Numerical Reliability: The rate of exponential decay provides a foundation for selecting simulation time horizons and numerical tolerances to ensure computational stability.
4. Performance guarantees: Exponential stability in control design means guaranteed transient performance bounds, which connect explicitly to engineering specifications[28].

The interplay between asymptotic and exponential stability and the related conditions under which the former implies the latter form an underlying theme of Sections 5 and 6 proper.[29]

4. Asymptotic Stability of Nonlinear Systems

4.1 Definition and Criteria

Asymptotic stability in nonlinear systems requires more than the existence of a limiting value. It requires a coherent framework that simultaneously guarantees both stability (boundedness) and attractivity (convergence). In nonlinear systems, these two properties generally do not imply one another, and this distinction is one of the major differences between nonlinear and linear stability theory[30].

Using the definitions introduced in Section 3.3.3, we now examine the principal criteria for asymptotic stability of nonlinear systems. These criteria may be broadly divided into three categories:

1. Lyapunov function based criteria,
2. Spectral criteria under structural assumptions,
3. Topological or geometrically motivated criteria.[31]

Lyapunov's second method (the direct method) provides the most fundamental and widely applicable criterion for asymptotic stability, and it will be studied in detail in Section 4.2. Before proceeding, however, it is useful to state several classical necessary conditions and clarify the relationship between local and global stability criteria.

Proposition 4.1 (Barbashin–Krasovskii Theorem)

Let V:Rn→RV : \mathbb{R}^n \to \mathbb{R}V:Rn→R be a continuously differentiable function satisfying:

V(0)=0,V(x)>0∀x≠0,V(0)=0, \qquad V(x)>0 \quad \forall x\neq 0,V(0)=0,V(x)>0∀x□=0,

V˙(x)=∇V(x)·f(x)≤−W(x),\dot{V}(x)=\nabla V(x)\cdot f(x)\leq W(x),V˙(x)=∇V(x)·f(x)≤−W(x),

for some positive definite continuous function WWW, V(x)→∞as|x|→∞.V(x)\to\infty \quad \text{as} \quad |x|\to\infty.V(x)→∞as|x|→∞.

Then the origin is globally asymptotically stable for system (3.1)[32].

Condition (iii), known as radial unboundedness or the properness condition, is precisely the requirement that V∈K∞V\in\mathcal{K}_\inftyV∈K∞ along rays from the origin, and it is this additional property that elevates local stability guarantees to global ones.[33]

Lyapunov's indirect method (linearization), developed further in Section 7.1, provides an important criterion linking nonlinear and linear theory. In particular, if the Jacobian matrix

A=Df(0)A=Df(0)A=Df(0)

has all eigenvalues with strictly negative real parts, then the origin is locally asymptotically stable for the nonlinear system (3.1)[24]. However, this result is local and provides no information regarding the size of the basin of attraction.

4.2 Lyapunov's Direct Method

Lyapunov's direct method also known as Lyapunov's second method is the central tool of nonlinear stability analysis. Its strength lies in establishing stability without requiring explicit solutions of the system trajectories.

The method proceeds by constructing a scalar function V(x)V(x)V(x), called a Lyapunov function, which serves as a generalized measure of the system's energy, and then showing that this energy decreases along trajectories.

Theorem 4.2 (Lyapunov Stability Theorem Local Version)

Let D⊆Rn\mathcal{D}\subseteq\mathbb{R}^nD⊆Rn be an open set containing the origin. Suppose there exists a continuously differentiable function

V:D→RV:\mathcal{D}\to\mathbb{R}V:D→R

such that

V(0)=0,V(x)>0∀x∈D\{0},(4.1)V(0)=0, \qquad V(x)>0 \quad \forall x\in\mathcal{D}\setminus\{0\}, \tag{4.1}V(0)=0,V(x)>0∀x∈D\{0},(4.1)

and

V˙(x):=∂V∂xf(x)≤0∀x∈D.(4.2)\dot{V}(x):=\frac{\partial V}{\partial x}f(x)\leq 0 \quad \forall x\in\mathcal{D}. \tag{4.2}V˙(x):=∂x∂Vf(x)≤0∀x∈D.(4.2)

Then the origin is Lyapunov stable. If additionally,

V˙(x)<0∀x∈D\{0},(4.3)\dot{V}(x)<0 \quad \forall x\in\mathcal{D}\setminus\{0\}, \tag{4.3}V˙(x)<0∀x∈D\{0},(4.3) then the origin is locally asymptotically stable.[34]

The proof proceeds by using the level sets

{x:V(x)≤c}\{x:V(x)\leq c\}{x:V(x)≤c}

as invariant regions for the flow. Positive definiteness of VVV guarantees that these sets are neighborhoods of the origin, while negative definiteness of V˙\dot{V}V˙ forces trajectories toward the equilibrium.

Although converse Lyapunov theorems guarantee the existence of Lyapunov functions whenever asymptotic stability holds, they generally provide no constructive method for finding such functions. This has motivated a large body of research on Lyapunov function construction techniques.[35]

LaSalle’s invariance principle significantly extends Lyapunov’s theorem by relaxing the requirement that V˙\dot{V}V˙ be strictly negative definite[36].

For polynomial systems, Lyapunov analysis is closely connected to sum of squares programming, which provides systematic computational procedures for searching for polynomial Lyapunov functions [37]. For systems with sector nonlinearities, Lur’e type Lyapunov functions provide useful stability certificates[38].

For gradient systems of the form

f(x)=−∇Φ(x),f(x)= \nabla \Phi(x),f(x)=−∇Φ(x),

the potential function itself naturally serves as a Lyapunov function, since

V˙=−|∇Φ|2≤0.\dot{V}= |\nabla \Phi|^2\leq 0.V˙=−|∇Φ|2≤0.

LaSalle’s invariance principle then guarantees convergence to the set of critical points of Φ\PhiΦ.

Recent advances in Lyapunov based analysis include:

- event triggered Lyapunov conditions for networked control systems,
- density function duals of Lyapunov functions,
- storage function methods for dissipative systems.

The density function approach provides a dual viewpoint in which global asymptotic stability is established through functions satisfying conditions complementary to those of classical Lyapunov functions.[39]

For non autonomous systems, the Lyapunov theorem must be modified.

Theorem 4.4 (Lyapunov Stability for Non Autonomous Systems)

Let

f:R+×D→Rnf:\mathbb{R}_+\times\mathcal{D}\to\mathbb{R}^nf:R+×D→Rn

satisfy the Carathéodory conditions. Suppose there exists a continuously differentiable function

V:R+×D→RV:\mathbb{R}_+\times\mathcal{D}\to\mathbb{R}V:R+×D→R

and class K\mathcal{K}K functions α1,α2\alpha_1,\alpha_2α1,α2 such that

α1(∣x∣)≤V(t,x)≤α2(∣x∣),(4.4)\alpha_1(|x|)\leq V(t,x)\leq \alpha_2(|x|), \tag{4.4}α1(∣x∣)≤V(t,x)≤α2(∣x∣),(4.4)

and

∂V∂t+∂V∂xf(t,x)≤−W(x),(4.5)\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) \leq W(x), \tag{4.5}∂t∂V+∂x∂Vf(t,x)≤−W(x),(4.5)

where WWW is a continuous positive definite function.

Then the origin is uniformly asymptotically stable.

The uniformity follows from the fact that α1\alpha_1α1, α2\alpha_2α2, and WWW are independent of time, ensuring that Lyapunov decay is uniform over all initial times.[24]

4.3 LaSalle's Invariance Principle

LaSalle's invariance principle is one of the most elegant and practically useful results in nonlinear dynamical systems theory. It provides asymptotic stability conclusions under significantly weaker assumptions than strict negative definiteness of V˙\dot{V}V˙.

Definition 4.1 (Positively Invariant Set)

A set

M⊆Rn\mathcal{M}\subseteq\mathbb{R}^nM⊆Rn

is positively invariant for system (3.1) if

x(0)∈M ⟹ x(t)∈M∀t≥0.x(0)\in\mathcal{M} \implies x(t)\in\mathcal{M} \quad \forall t\geq 0.x(0)∈M⟹x(t)∈M∀t≥0.

Theorem 4.5 (LaSalle's Invariance Principle)

Let

Ω⊆D\Omega\subseteq\mathcal{D}Ω⊆D

be a compact positively invariant set for system (3.1). Let

V:D→RV:\mathcal{D}\to\mathbb{R}V:D→R

be continuously differentiable and satisfy

V˙(x)≤0∀x∈Ω.\dot{V}(x)\leq 0 \quad \forall x\in\Omega.V˙(x)≤0∀x∈Ω.

Define

E={x∈Ω:V˙(x)=0}.(4.6)\mathcal{E} = \{x\in\Omega:\dot{V}(x)=0\}. \tag{4.6}E={x∈Ω:V˙(x)=0}.(4.6)

Let M\mathcal{M}M denote the largest positively invariant subset of E\mathcal{E}E. Then every trajectory starting in Ω\OmegaΩ converges to M\mathcal{M}M as t→∞t\to\inftyt→∞.

This principle states that trajectories converge not necessarily to a point, but to the largest invariant set contained within the set where V˙=0\dot{V}=0V˙=0. If this invariant set consists only of the equilibrium {0}\{0\}{0}, asymptotic stability follows immediately.

Suppose VVV is positive definite and V˙≤0\dot{V}\leq 0V˙≤0, but V˙\dot{V}V˙ vanishes on a set larger than {0}\{0\}{0}. One then analyzes the dynamics restricted to this set. If the only trajectory remaining there for all time is the trivial solution x(t)≡0x(t)\equiv 0x(t)≡0, then the origin is asymptotically stable.

Example 4.1 (Damped Pendulum)

Consider the damped pendulum equation

θ¨+bθ˙+sinθ=0,b>0.(4.7)\ddot{\theta} + b\dot{\theta} + \sin\theta = 0, \qquad b>0. \tag{4.7}θ¨+bθ˙+sinθ=0,b>0.(4.7)

Let the state be

x=(θ,θ˙).x=(\theta,\dot{\theta}).x=(θ,θ˙).

The total mechanical energy is

V(x)=12θ˙2+(1−cosθ).(4.8)V(x) = \frac{1}{2}\dot{\theta}^2 + (1 \cos\theta). \tag{4.8}V(x)=21θ˙2+(1−cosθ).(4.8)

This function is positive definite near the origin and satisfies

V˙(x)=−bθ˙2≤0.\dot{V}(x) = b\dot{\theta}^2 \leq 0.V˙(x)=−bθ˙2≤0.

The set

E={x:θ˙=0}\mathcal{E} = \{x:\dot{\theta}=0\}E={x:θ˙=0}

contains all states with zero angular velocity. On this set,

θ¨=−sinθ=0,\ddot{\theta} = \sin\theta = 0,θ¨=−sinθ=0,

which implies

θ=kπ,\theta=k\pi,θ=kπ,

for integers kkk. Near the origin, the only equilibrium is

θ=0,θ˙=0.\theta=0,\qquad \dot{\theta}=0.θ=0,θ˙=0.

Therefore,

M={0},\mathcal{M}=\{0\},M={0},

and the origin is asymptotically stable by LaSalle's principle.

LaSalle's principle has since been extended to adaptive control, non autonomous systems, infinite dimensional systems, and non smooth dynamics such as Filippov differential inclusions and hybrid systems[40], [41], [42].

4.4 Stability of Autonomous Systems

Autonomous systems, systems in which the vector field has no explicit time dependence, possess structural properties that simplify stability analysis considerably[43], [44].

Three important properties are:

1. time translation invariance of trajectories,
2. existence of well defined ω\omegaω limit sets,
3. applicability of LaSalle's principle in full generality.

Definition 4.2 (ω\omegaω Limit Set)

For a trajectory ϕ(t,x0)\phi(t,x_0)ϕ(t,x0) of system (3.1), the ω\omegaω limit set of x0x_0x0 is

ω(x0)={y∈Rn:∃ tk→∞ such that ϕ(tk,x0)→y}.(4.9)\omega(x_0) = \left\{ y\in\mathbb{R}^n : \exists\, t_k\to\infty \text{ such that } \phi(t_k,x_0)\to y \right\}. \tag{4.9}ω(x0)={y∈Rn:∃tk→∞ such that ϕ(tk,x0)→y}.(4.9)

The ω\omegaω limit set consists of all accumulation points of the trajectory as t→∞t\to\inftyt→∞. For bounded trajectories of autonomous systems, ω\omegaω limit sets are nonempty, closed, connected, and positively invariant.

For two dimensional systems, the Poincaré–Bendixson theorem states that if a trajectory remains inside a compact region containing no equilibria, then it must converge to a periodic orbit.

In higher dimensional systems, ω\omegaω limit sets may include:

- equilibria,
- periodic orbits,
- homoclinic and heteroclinic orbits,
- quasi periodic motions,
- chaotic attractors.

Center manifold theory plays an important role in asymptotic stability analysis when the linearization possesses eigenvalues with zero real parts[45].

Theorem 4.6 (Center Manifold Theorem)

Suppose

A=Df(0)A=Df(0)A=Df(0)

has:

- nsn_sns eigenvalues with negative real parts,
- nun_unu eigenvalues with positive real parts,
- ncn_cnc eigenvalues with zero real parts,

with

ns+nu+nc=n.n_s+n_u+n_c=n.ns+nu+nc=n.

Then there exist locally invariant manifolds:

- WsW^sWs (stable),
- WuW^uWu (unstable),
- WcW^cWc (center),

of dimensions nsn_sns, nun_unu, and ncn_cnc, respectively, tangent to the associated eigenspaces at the origin.

Furthermore, asymptotic stability of the full system is equivalent to asymptotic stability of the reduced dynamics on the center manifold WcW^cWc[46].

Center manifold reduction has become an important analytical tool in mechanics, control theory, and mathematical biology.[46]

4.5 Stability under Perturbations

The stability properties of an idealized mathematical model are practically meaningful only if they persist under perturbations and modeling uncertainties.

Consider the perturbed system

x˙=f(x)+g(t,x),(3.3)\dot{x}=f(x)+g(t,x), \tag{3.3}x˙=f(x)+g(t,x),(3.3)

where f(0)=0f(0)=0f(0)=0 and ggg represents a perturbation that may be time varying, uncertain, or induced by modeling errors.

The central question is: under what conditions on ggg does asymptotic stability of the nominal system

x˙=f(x)\dot{x}=f(x)x˙=f(x)

imply boundedness or asymptotic stability of the perturbed system?

Theorem 4.7 (Total Stability / Ultimate Boundedness)

Suppose the origin of the nominal system is globally asymptotically stable, and assume

∣g(t,x)∣≤δ|g(t,x)|\leq \delta∣g(t,x)∣≤δ

for all t≥0t\geq 0t≥0 and x∈Dx\in\mathcal{D}x∈D, where δ>0\delta>0δ>0 is sufficiently small.

Then every solution of the perturbed system is ultimately bounded; that is, there exists a compact set

B∋0\mathcal{B}\ni 0B∋0

such that every trajectory enters B\mathcal{B}B in finite time and remains there thereafter.

A more powerful framework for robustness analysis is Input to State Stability (ISS).

A system

x˙=f(x,u)\dot{x}=f(x,u)x˙=f(x,u)

is ISS with respect to the input u∈L∞u\in L^\inftyu∈L∞ if there exist functions

β∈KL,γ∈K,\beta\in\mathcal{KL}, \qquad \gamma\in\mathcal{K},β∈KL,γ∈K,

such that

∣x(t)∣≤β(∣x(0)∣,t)+γ(sup0≤s≤t∣u(s)∣),(4.10)|x(t)| \leq \beta(|x(0)|,t) + \gamma\left(\sup_{0\leq s\leq t}|u(s)| \right), \tag{4.10}∣x(t)∣≤β(∣x(0)∣,t)+γ(0≤s≤tsup∣u(s)∣),(4.10)

for all t≥0t\geq 0t≥0.

ISS unifies:

- asymptotic stability (when u≡0u\equiv 0u≡0),
- bounded input bounded state stability,
- small gain analysis for interconnected systems.

Modern ISS theory has expanded to include:

- integral ISS (iISS),
- input to output stability (IOS),
- ISS for PDEs and infinite dimensional systems,
- ISS for time delay systems,
- hybrid systems,
- stochastic systems.

The Lyapunov characterization of ISS through ISS Lyapunov functions satisfying dissipation inequalities remains the principal tool for verifying ISS properties in concrete systems.

Another important perturbation framework concerns structured uncertainty in the frequency domain, studied through μ\muμ analysis and H∞H_\inftyH∞ methods. Extensions of L2L_2L2 gain analysis and incremental stability methods to nonlinear systems continue to be active areas of research.[47]

5. Exponential Stability of Nonlinear Systems

5.1 Definition and Mathematical Formulation

Exponential stability (introduced through as Definition 3.7) defines the most rigorous well known type of quantitative asymptotic characteristics sought in practice. This definition that the state norm can be upper bounded by a decaying exponential function of time (for a certain region coordinate with respect to initial conditions) with exponent independent of x0 has clean theoretical completeness and translates directly into practice. We turn to a deeper, mathematical theory of exponential stability.[48]

The local exponential stability condition (equation (3.11)) for the autonomous system in article 3.1 can then be equivalently expressed based on using a Lyapunov function:

Let us restate Example 5.1Theorem 5.1 (Lyapunov Characterization of Exponential Stability). System (3.1) is locally exponentially stable at the origin if and only if there exist a continuously differentiable function V : ℝn → ℝ and constants c_1, c_2, c_3, r > 0 such that for every x ∈ B(0,r):

V(x) \geq c_1 |x|^2, \hspace{5cm} V(x) \leq c_2 |x|^2. (5.1)

\dot{V}(x) \leq c_3 |x|^2. \tag{5.2}

The quadratic conditions (5.1) and (5.2) are of exponential type, which differentiates the exponential stability from only qualitative asymptotic stability. Based on (5.1) and (5.2), we can proceed directly to the exponential decay estimate: differentiating V along trajectories yields

where \frac{d}{dt}V(x(t)) \leq c_3 |x(t)|^{2}\leq \frac{c_{3}}{c_{2}}V(x(t)),(5.3)

which by Gronwall's inequality yields:

V(x(t)) \leq V(x(0)) e^{ (c_3/c_2) t}. \tag{5.4}

Together with the left inequality in (5.1), gives:

\begin{equation} |x(t)| \leq \sqrt{\frac{c_2}{c_1}}, |x(0)|, e^{ (c_3/(2c_2)) t}, (5.5) \end{equation}

which is exactly the corresponding exponential stability estimate (3.11) with $c = \sqrt{c_2/c_1}$ and $\lambda = c_3/(2c_2)$.

In contrast, for the case of exponential stability, there is a converse Lyapunov theorem which ensures existing of a C^1 Lyapunov function satisfying (5.1)–(5.2). In the construction of this converse Lyapunov function, the flow of system over time is integrated, a method that guarantees theoretical existence but what not directly computationally tractable in general.

We require with respect to uniformity over any initial time in order for the nonautonomous system (3.2) to be exponentially stable:

Definition 5.1 (Uniform Exponential Stability). System (3.2) is uniformly exponentially stable (UES) if $\exists c, \lambda, \delta_0 > 0$ such that $\forall t_0 \geq 0$:

\tag{3.21} | x (t) −x (t_0)| 0$, and is thus exponentially stabl;e. This equivalence is immediate from the Jordan form and the spectral mapping theorem[24]. No such general equivalence exists in the case of nonlinear systems[49].

Nonetheless, under additional structural conditions on the nonlinearity, exponential stability can be assured. The main result in this direction is:

Theorem 5.2 (Exponential stability by linear approximation). Let us assume that (3.1) has an equilibrium at the origin, f is continuously differentiable in a neighborhood of the origin, and A ¼ Df(0) has all eigenvalues with strictly negative real parts. Now if the origin is locally exponentially stable.

Theorem 5.2 is a refinement of the indirect Lyapunov method and asserts that whenever the linearization is asymptotically stable (i.e., Hurwitz), then so is the nonlinear system, i.e., it is locally exponentially stable[24]. The proof entails demonstrating that the Lyapunov equation $A^T P + PA = I$ has a positive definite solution, and that $V(x) = x^T P x$ is locally V (temporal term) margin (5.1)–(5.2).

For a more global resolution, tougher conditions are necessary. An important class is that of one sided Lipschitz systems:

Definition 5.2 (One Sided Lipschitz Condition). Let $\varphi : \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a Plancherel differentiable function with $\partial_{1}\varphi,,\cdots,\partial_{k}g(W) = g(X)$ f satisfy one sided Lipschitz condition at z_0: In other words, if there exists the constant $\rho\in \mathbb{R}$ such that:

⟨f(x) − f(y), x − y⟩ ≤ ρ |x − y|^2 ∀x,y ∈ Rn. (5.7)[50]

Conversely, if $f(0) = 0$ and (5.7) holds with $\rho 0$ (this condition is satisfied if A is Hurwitz), and $|g(x)| \leq L|x|^2$, for all $(x | x|\leq r)$, then, for small enough value of r:

\dot{V}(x) \leq \lambda_{min}(Q)| x|^2 + 2|P| \\cdot L | x|^3 \leq \frac{\lambda _{min}(Q)}{2}| x|^2,tag(5.9)

achieving exponential stability with rate $\lambda = \lambda_{\min}(Q)/(4\lambda_{\max}(P))$.

The results are obtained based on the quadratic Lyapunov function for globally Lipschitz nonlinear systems, which develop global exponential stability conditions. Let the function f satisfy the growth condition {|f(x)|≤L|x|,} for all x∈ℝn and some L>0. Global exponential stability can then be certified by a quadratic Lyapunov function solving the LMI:

A^T P + PA + 2LP 0$. The homogeneity framework has already been generalized to non smooth systems, and can be used for synthesis of homogeneous control laws with desired finite time or exponential convergence[51].

Integral Quadratic Constraints. Integral Quadratic Constraints (IQCs) are a natural and generalization of the quadratic Lyapunov approach, in which frequency domain inequalities characterize the behavior of nonlinear operators. Stability conditions based on linear matrix inequalities (LMI) were initially presented for systems exhibiting an IOQ but have since been extended to operate within the IQC framework using multiplier methods and subsequently applied to the analysis of feedback systems with Lipschitz nonlinearities.[52]

5.4 Uniform and Nonuniform Exponential Stability

This enables examination of the differences between uniform and non uniform exponential stability, which for autonomous systems (where the two notions coincide) is less practically relevant but becomes vital when considering time varying or non autonomous systems.[53]

Non Uniform Exponential Stability. For the non autonomous system (3.2), its equilibrium is said to be non uniformly exponentially stable if there are functions $c(t_0) > 0$ and a constant $\lambda > 0$ such that:

Inequality (5.11) holds for any t if |x (t)| ≤ c(t_0) e−λ(t−t 0) |x (t_0)| ∀ t ≥ t 0.

such that $c(t_0)$ may depend on t_0, and λ is independent of $t_0$contribute to expand but can take arbitrary output. Non uniform exponential stability (Definition \ref{def:non uniform}) is a much weaker notion than uniform exponential stability (Definition \ref{def:uniform}), as the overshoot constant $c(t_0)$ may tend to infinity as $t_0\to \infty$, although the decay rate λ is fixed. As a result, non uniformly exponentially stable system might have arbitrarily big initial transients for large enough t_0 making the robustness of response to perturbations very limited.[54]

Non uniform exponential stability and robustness have been studied in detail in the literature. One significant conclusion is that asymptotic stability notion: non uniform exponential stability should not be confused with ISS, i.e. robustness to perturbations (in any general sense), and the uniform notion is practically better than the non uniform one.[55]

Stability Theory of Uniform Exponential Stability and Linear. For linear time varying system $\dot{x} = A(t)x$, uniform exponential stability of $x=0$ is equivalent to the existence of symmetric positive definite matrix valued function $P (t)$ that satisfies the differential Riccati inequality:

(5.12) dotPP(t)+PA(t)+AT (t)P(t)=μ P(t) 0$. This condition, which is a time varying version of the Lyapunov equation, offers a systematic computational criterion for uniform exponential stability. Extensions to nonlinear systems can be obtained by imposing that the quadratic Lyapunov function condition (5.1)–(5.2) hold uniformly in time.[56]

Exponential Stability for PDEs. For infinite dimensional systems modeled by PDEs, we usually define exponential stability as follows in terms of the norm defined on a Hilbert or Banach space of states:

In such cases, the Green's inequality implies that it holds for all initial data, namely |u(·,t)|H≤Ce λt|u(·,0)|H,(5.13)

for constants $C, \lambda > 0$ In the case of PDEs, exponential stability uses semigroup theory (that is, C_0 semigroups and their spectral properties. The spectrum determined growth property is valid for analytic semigroups (generated by sectorial operators), namely, the exponential decay rate can be read off from the spectrum of the generator. This property may not hold when we do not have an analytic semigroup, so that a direct Lyapunov functional study might be required. Recent works include results on exponential stability for some nonlinear hyperbolic PDEs of traffic flow and irrigation channels.[57]

6. Analytical Methods for Stability Analysis

6.1 Linearization Techniques

Linearization remains one of the most fundamental analytical tools for investigating the local stability properties of nonlinear systems. The technique exploits the principle that sufficiently close to an equilibrium, the behavior of a nonlinear system is well approximated by its linear part the Jacobian evaluated at the equilibrium.[58]

Lyapunov's First Method (Indirect Method). For system (3.1) with f continuously differentiable and $f(0) = 0$, define the Jacobian $A = \frac{\partial f}{\partial x}\big|{x=0}$. The linearized system is: \dot{y}(t) = Ay(t). \tag{6.1}

Lyapunov's first method establishes the following:

Theorem 6.1 (Linearization Stability Theorem). (i) If all eigenvalues of A have strictly negative real parts (i.e., A is Hurwitz), the origin of the nonlinear system (3.1) is locally exponentially stable. (ii) If at least one eigenvalue of A has strictly positive real part, the origin is unstable. (iii) If some eigenvalues have zero real parts and none have positive real parts, the linearization is inconclusive the nonlinear terms determine stability.

Case (iii) the center case is the most delicate and requires the center manifold theory described in Section 4.4, normal form analysis, or higher order Lyapunov analysis. The center case arises, for example, at bifurcation points where a Hopf bifurcation occurs, and its analysis is central to understanding the onset of oscillatory behavior in biological and physical systems.[59] Computational implementation of linearization stability analysis proceeds via computation of the eigenvalues of A, which can be performed efficiently for moderate dimensional systems using standard linear algebra algorithms. For systems arising from spatial discretization of PDEs, however, the Jacobian may be large and sparse, requiring iterative eigenvalue solvers.[60] Hartman–Grobman Theorem. A deeper result connecting the topological structure of the nonlinear flow near a hyperbolic equilibrium (one where all eigenvalues of A have nonzero real parts) to that of the linearized flow is the Hartman–Grobman theorem:

Theorem 6.2 (Hartman–Grobman Theorem). If $x = 0$ is a hyperbolic equilibrium of (3.1), then there exists a homeomorphism h defined in a neighborhood of the origin that maps trajectories of the nonlinear system (3.1) to trajectories of the linearized system (6.1), preserving the direction of time.[61]

The Hartman–Grobman theorem establishes that near hyperbolic equilibria, the qualitative dynamics of the nonlinear system are topologically equivalent to those of its linearization. While this is a purely topological result and provides no quantitative bounds, it has important implications for the classification of equilibrium types in two dimensional systems (nodes, spirals, saddle points) and for understanding the local geometry of stable and unstable manifolds.[62]

Feedback Linearization. In control engineering, exact feedback linearization transforms a nonlinear system into a linear one through a nonlinear coordinate change and feedback. For an affine nonlinear control system:

$$\dot{x} = f(x) + g(x)u, \tag{6.2}$$

exact feedback linearization (when applicable) finds a diffeomorphism $z = T(x)$ and a control law $u = \alpha(x) + \beta(x)v$ such that the closed loop system in the new coordinates is the linear system $\dot{z} = Az + Bv$. Stability of the linearized system then implies stability of the original nonlinear system in the domain of the diffeomorphism. The applicability of this approach is governed by involutivity conditions on the Lie algebra generated by f and g conditions that may fail for general nonlinear systems.[63]

6.2 Lyapunov Function Construction Methods

The direct method of Lyapunov provides necessary and sufficient conditions for stability (through converse theorems), but its practical application hinges entirely on the availability of a suitable Lyapunov function. The systematic construction of Lyapunov functions for nonlinear systems is therefore one of the most practically important and theoretically challenging problems in stability analysis.

Variable Gradient Method. The variable gradient method seeks a Lyapunov function by assuming a parameterized form for its gradient $\nabla V(x) = \phi(x)$, where ϕ is chosen to satisfy the integrability condition $\partial \phi_i / \partial x_j = \partial \phi_j / \partial x_i$, and then requiring $\dot{V}(x) = \phi(x)^T f(x) < 0$. This reduces the Lyapunov function search to an algebraic system of equations, which may be solvable for specific nonlinear structures. The method is effective for systems of dimension two or three but becomes computationally intractable in higher dimensions.

Sum of Squares (SOS) Programming. The SOS approach provides a systematic computational framework for searching polynomial Lyapunov functions via semidefinite programming. A polynomial $p(x)$ is a sum of squares if $p(x) = \sum_i q_i(x)^2$ for some polynomials q_i; sum of squares polynomials are clearly nonneg definite, and verifying the SOS property reduces to a semidefinite program (SDP) that can be solved efficiently.

The SOS framework reformulates the Lyapunov function search as follows: given the polynomial system $\dot{x} = f(x)$ with polynomial f, find a polynomial $V(x)$ such that $V(x) \varepsilon |x|^2$ and $ \dot{V}(x) \varepsilon |x|^2$ are both SOS polynomials for some $\varepsilon > 0$. This search can be encoded as an SDP and solved using standard solvers. The principal limitation of SOS methods is their restriction to polynomial (or rational) systems and their polynomial growth in computational cost with system dimension the number of SOS variables grows as $O(n^{2d})$ for degree d, making the approach impractical for dimensions beyond approximately $n = 15$ 20 for moderate degree.

Neural Network Lyapunov Functions. The application of neural networks to Lyapunov function synthesis has emerged as one of the most active research frontiers in stability analysis. The approach treats the Lyapunov function as a parameterized class of neural networks and optimizes the parameters to satisfy the Lyapunov conditions. Several frameworks have been proposed:

Verifiable neural Lyapunov functions train a neural network $V\theta(x)$ using a loss function that penalizes violations of the Lyapunov conditions, combined with formal verification of the trained network using Satisfiability Modulo Theories (SMT) solvers or interval arithmetic to certify the conditions over a domain. Neural Lyapunov control simultaneously learns a neural Lyapunov function and a stabilizing controller, with formal verification providing closed loop stability guarantees. The approach has demonstrated applicability to systems in dimensions up to approximately $n = 10$ with significant nonlinearity.[64]

Counterexample guided synthesis iteratively alternates between synthesis (training the neural network) and verification (checking the Lyapunov conditions), using counterexamples found by the verifier to improve subsequent synthesis rounds. This approach provides a systematic convergence mechanism and has been applied to autonomous systems, control systems, and hybrid systems.[65]

The neural network approach has the significant advantage of not being restricted to polynomial systems and of scaling better with dimension than SOS methods. However, it introduces challenges related to the soundness of the verification step, the completeness of the approach (i.e., whether it will succeed when a Lyapunov function exists), and the interpretability of the resulting certificates.[66]

Control Lyapunov Functions and Backstepping. For systems with control inputs, the concept of a control Lyapunov function (CLF) provides a unified framework for stabilization and Lyapunov function construction. A CLF is a positive definite function V for which there exists, at every nonequilibrium state, a control input u that makes $\dot{V}$ negative definite. The existence of a CLF is equivalent to smooth stabilizability, and explicit stabilizing control laws can be constructed from a CLF via universal formulas for nonlinear stabilization.[67]

Backstepping is a systematic recursive procedure for constructing Lyapunov functions and stabilizing controllers for triangular (strict feedback) nonlinear systems. At each step of the recursion, a Lyapunov function for a subsystem is extended to a Lyapunov function for a larger subsystem by incorporating a quadratic term penalizing the deviation from a virtual control signal. Backstepping has been extended to adaptive systems, output feedback systems, and stochastic systems.[68]

6.3 Fixed Point Theorems

Fixed point theorems, while less directly associated with stability analysis than Lyapunov methods, provide powerful tools for establishing the existence and uniqueness of equilibria, as well as for analyzing contraction based stability.[69]

Banach's Fixed Point Theorem (Contraction Mapping Theorem). Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping, i.e., $d(Tx, Ty) \leq k , d(x, y)$ for some $k \in (0, 1)$ and all $x, y \in X$. Then T has a unique fixed point $x^* \in X$, and the iteration $x_{n+1} = Tx_n$ converges to x^* at a geometric rate.

In the context of differential equations, the contraction mapping theorem is applied both to prove existence and uniqueness of solutions (via the Picard iteration) and to analyze exponential stability through contraction of the flow map. Specifically, if the flow $\phi(t, \cdot)$ of system (3.1) is a contraction on $\mathcal{D}$ for each $t > 0$, the system is exponentially stable. This observation forms the basis of contraction analysis, which studies exponential stability through the contraction properties of the Jacobian of the flow: \frac{d}{dt}|J(t)| \leq \lambda |J|, \tag{6.3} where $J(t) = \frac{\partial \phi}{\partial x_0}(t, x_0)$ is the fundamental matrix solution. Contraction analysis has been applied to observer design, synchronization of coupled oscillators, and stability of neural networks.

Brouwer and Schauder Fixed Point Theorems. For establishing the existence (but not uniqueness) of equilibria, the topological fixed point theorems of Brouwer (finite dimensional) and Schauder (infinite dimensional) are employed. Brouwer's theorem states that any continuous map from a compact convex subset of $\mathbb{R}^n$ to itself has a fixed point. Schauder's theorem extends this to compact continuous maps on Banach spaces. These results are used in the stability analysis of functional differential equations, integral equations, and boundary value problems.[70]

6.4 Comparison Principles

Comparison principles provide a powerful technique for bounding solutions of nonlinear differential equations by comparing them to solutions of simpler reference systems.

Gronwall–Bellman Inequality. The foundational comparison result is the Gronwall–Bellman inequality: if $u(t) \leq \alpha(t) + \int_{t_0}^t \beta(s) u(s) , ds$ for nonneg functions u, α, β, then: u(t) \leq \alpha(t) + \int_{t_0}^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r) , dr\right) ds. \tag{6.4}

In stability analysis, the Gronwall inequality is used to derive exponential bounds on solution norms from differential inequalities of the form $\frac{d}{dt}|x(t)|^2 \leq c|x(t)|^2 + \varepsilon$, which arise naturally from Lyapunov analysis of perturbed systems.[71]

Differential Inequalities and Comparison Systems. More generally, suppose $\dot{v}(t) \leq h(t, v(t))$ and $\dot{w}(t) = h(t, w(t))$ with $v(t_0) \leq w(t_0)$. Under appropriate monotonicity conditions on h, the comparison principle guarantees $v(t) \leq w(t)$ for all $t \geq t_0$. This technique transforms the stability analysis of a complex nonlinear system into the stability analysis of a simpler scalar or low dimensional comparison system.[72]

Comparison principles have been extensively applied in the stability analysis of large scale interconnected systems, neural networks, biological models, and time delay systems. The M matrix theory provides algebraic conditions for the stability of comparison systems arising from interconnected networks, linking network structure (through the graph Laplacian or coupling matrix) to overall system stability.[73]

7. Numerical and Computational Approaches

7.1 Numerical Stability Analysis Methods

Although analytical methods are able to offer exact stability guarantees in some neighborhood of attraction due to their specific domain of applicability, many practical nonlinear systems either cannot be treated analytically (complexity, high dimensionality and lack of closed form structure) or do inherently support higher order behavior via dependence upon integral arguments rendering such detail impossible (for example as with motion control). This has led to the development of a numerical stability analysis toolbox, which mainly aims at providing verification and validation tools for stability properties in concrete applications.[74]

The numerical analysis of stability can be broadly summarized as three inter related tasks: (i) the numerical computation and classification of equilibria, (ii) the numerical estimation of regions of attraction, and (iii) the numerical computation or verification of Lyapunov functions.

Numerical Computation of Equilibria. The equilibria of the system (3.1) are the solutions of this algebraic equation $f(x^*) = 0$. Homotopy continuation methods applied to polynomial vector fields can find all the solutions of a system of polynomials, i.e., partial balance point equations in $\mathbb{C}^n$ and later filter for real solutions giving us the complete enumerations of equilibria. When dealing with general nonlinear systems, locally convergent methods for isolating equilibria include Newton–Raphson iteration and its variants; while globally convergent methods exist in the form of interval Newton arithmetic.[75], [76] Next, we classify each of the equilibria as stable, unstable or saddle type using eigenvalue analysis of the Jacobian evaluated at the equilibrium.[77]

Region of Attraction Estimation. Estimating the region of attraction (basin of attraction) $\mathcal{R}_A$ around a locally asymptotically stable equilibrium is a highly practical problem for safety critical applications, it is important to know within what regions initial conditions guarantee convergence. In the classical framework, sublevel sets ${x : V(x) \leq c}$ of a Lyapunov function are used as inner estimates for $\mathcal{R}_A$; if $\dot{V}(x) < 0$ on the sublevel set then that region is positively invariant and contained in $\mathcal{R}_A$. The bound is tightest within the Lyapunov framework as it maximizes the size of the certified sublevel set over choice of Lyapunov function V and level c.

This optimization can be implemented through SOS programming, which gives a systematic computationally tractable way to find optimal solutions [2]. This problem can be solved as the following SOS optimization:

\begin{equation} \max_{V,c}\; \text{ vol}({x : V(x) \leq c})\\\quad\text{s.t.}\\ {x : V(x)\le c}\subseteq {x:\dot V(x)\le 0}.\\tag{7.1} \end{equation}

This can be loosened to an SDP using Positivstellensatz certificates[78]. The region of attraction estimation problem is solved by alternative approaches based on occupation measures through a hierarchy of SDP relaxations, which are proved to converge from the outside to the exact region of attraction.[79]

Numerical Lyapunov Function Computation. In addition to SOS and methods based on neural networks (Section 6.2), other numerical Lyapunov function computation methods were implemented. Zubov's approach describes the exact basin of attraction as a level set of an information driven Partial Differential Equation (PDE) known as the Zubov PDE, which when solved numerically, produces the Lyapunov function and the actual region of attraction. Many numerical solvers for the Zubov PDE exist based on finite differences, finite elements or level set methods (e.g.).

Mesh based approaches use linear programming at the mesh vertices to establish if a piecewise linear or piecewise polynomial Lyapunov function can be computed on a triangulation of the state space[80]. These approaches yield precise stability certificates for nonlinear systems that remain constrained within a certain domain and have found applications in the CPA (Continuous Piecewise Affine) framework.[80]

7.2 Simulation Techniques

Dynamic modelling and numerical simulation of differential equations provide the most commonly used practical tool to study stability behaviour, although establishing limits on dynamic models treated in isolation from other aspects does not amount to rigorous stability proofs[81].

ODE Solvers and Stiffness. The most commonly used dynamical simulation methods are standard numerical ODE solvers such as explicit Runge–Kutta (e.g., RK4, Dormand–Prince), implicit methods (e.g., backward Euler, trapezoidal rule) and multistep methods (Adams–Bashforth, BDF).[82] The stiffness of the system determines which solver you can use: stiff systems, for instances when the Jacobian eigenvalues span a wide range of magnitudes, require implicit solvers to operate without due prohibitive restrictions on the step size. As an example, stiffness occurs in systems with fast and slow dynamics like enzyme kinetics (Briggs Haldane mechanism), power systems with electromagnetics and electromechanical time scales or chemical reaction networks.[83]

For two dimensional systems, phase portrait visualizations give us direct geometric insight into stability behavior. When $n = 2$, the flow lines of (3.1) can be drawn in the (x_1, x_2) plane: equilibria, limit cycles, separatrices and basins of attraction. Standard mathematical software packages (MATLAB, Python/SciPy, Julia/DifferentialEquations) implement numerical computation of phase portraits. jl) and it stays a first look tool for stability assessment.[84]

In systems that are higher dimensional ($n \geq 3$), one relies on projection methods, Poincaré sections and invariant manifold visualizations to understand the phase space structure. The stable and unstable manifold numerical computation methods which are key in understanding the global structure of phase space and the attraction basin boundaries can be approached also through the parameterization methods or boundary value problem formulations.[85]

Monte Carlo and Probabilistic Methods. Statistical methods for assessing stability based on running a set of simulations from initial conditions chosen randomly, and measuring the empirical distribution of convergence. Although they do not offer rigorous guarantees, Monte Carlo based methods can rapidly estimate the fraction of state space contained within the region of attraction, identify regions near basin boundaries in which stability is uncertain, and perform sensitivity analyses on parameter perturbations.[86]

Event Driven Simulation. Event driven simulation frameworks keep track of the times at which state trajectories cross switching surfaces, process the associated discrete transitions, and restart continuous integration for hybrid systems and systems with discontinuous dynamics. MATLAB/Simulink and the Julia DifferentialEquations. jl package offer to you event driven simulation implementations which use adaptive step size control that preserves numerical precision over regions with discontinuities.

Software tools for stability analysis

To aid in the computation of stability, a rich software ecosystem ranging from general purpose environments for mathematical computing to specialized toolboxes for performing stability analysis has emerged.

Control oriented stability analysis is still mainly done in dominant platforms such as MATLAB and Simulink. EVA functions in the Control System Toolbox for eigenvalue analysis, Lyapunov equation solvers (lyap, dlyap), and LMI based stability verification. Add $H_\infty and μ \mu$ synthesis. Robust Control Toolbox Add on to the Robust Control Toolbox MATLAB has a symbolic toolbox that allows you to obtain analytical Lyapunov functions in small dimensional systems. Simulink offers graphical block diagram modeling with ODE solvers and linearization capabilities.[87]

SOSTOOLS is a MATLAB toolbox for sum of squares programming with high level interface to SOS decomposition and stability certification. Users give a specification as polynomial Lyapunov function templates and SOS constraints, SOSTOOLS translates these specifications into SDPs to be solved by a numerical solver under the hood. Extensions are proposed, including ßDSOS (Diagonally Dominant SOS) and ßSDSOS (Scaled Diagonally Dominant SOS) relaxations for higher scalability.[88]

Julia has continued growing as a platform for numerical stability analysis, with packages such as DifferentialEquations. Symbolics.jl Performance Friendly ODE Simulation in Julia Jeremy and several students from his lab built populationjl, a package for using jl for symbolic computation, as well as JuMP. Mathematical programming (including SDP formulations) in jl This is particularly well suited for high dimensional simulation and Monte Carlo stability assessment because of the performance characteristics of the Julia ecosystem.[89]

Python has the SciPy ecosystem for ODE simulation and linear algebra, the CVXPY library for convex programming as well as SDPs, and an interface to the dreal SMT solver to reason about non linear arithmetic formulas that is what we need in order to formally verify neural Lyapunov functions!!

Specialized Tools. CORA enables reachability analysis and safety verification of nonlinear systems using Lyapunov based invariant set computation with polynomial zonotopes. SpaceEx addresses hybrid system reachability.[90] LROA is a toolbox for estimating the region of attraction in MATLAB as well as SOS based methods for generating ROAs. PRISM and STORM focus on probabilistic model checking that is relevant to stability analysis for stochastic systems.[91]

7.4 Challenges in Numerical Verification

These analyses are empowered by a wealth of analytical and computational tools for numerical stability analysis, however there are still some fundamental challenges that restrict the applicability and verifiability of computational stability verification.[92]

Curse of Dimensionality. Full numerical stability analysis methods tend to scale poorly with the dimension of the system. SOS methods scale as $O(n^{2d})$ in the number of SDP variables with polynomial degree d; mesh based Lyapunov methods need exponentially growing mesh resolution in dimension, reachability methods suffer similar trouble. Scalability limitations would then constrain rigorous numerical stability certification to small to mid dimensional systems (for instance $n \leq 15$ 20 for SOS, $n \leq 6$ 8 for mesh based methods), with high dimensional systems common in multi robot systems, network dynamics and discretized PDEs [1] remaining overwhelmingly out of reach of rigorous numerical methods.[66]

Non Smooth and Hybrid Systems. Most numerical ODE solvers assume a smooth vector field, while non smooth systems (friction, impacts, switching) require specialized treatment. Computational stability tools are mainly based on smooth Lyapunov theory, while non smooth Lyapunov theory is not yet well integrated within availability. As we highlighted in Section 7, the numerical detection and treatment of Zeno behavior (i.e., an infinite number of events occurring in finite time in hybrid systems) remains a challenging open problem.[93]

Verification of Neural Network Certificates. Lyapunov functions based on neural networks need formal verification that the trained network respects the Lyapunov conditions over the asserted domain. The property satisfaction over a set by a neural network is an instance of the verification problem for that class of problems, commonly referred to as NN verification, which is NP complete in general[94]. While our approach is tractable for the many network architectures and types of properties, scalable verification of large networks or in high dimensional domains remains an ongoing research problem.

Numerical Soundness. One potential reason is that the use of floating point arithmetic, which can introduce rounding errors so nonincreasing Lyapunov certificates computed numerically may satisfy the Lyapunov conditions in the presence of a rigorous check using exact arithmetic. While this concern is addressed by using rigorous interval or verified numerical methods, it incurs enormous computational cost. This tradeoff is often a challenge in numerical efficiency versus rigorous soundness for computational stability analysis.[95]

8. Applications of Stability Theory

8.1 Engineering Systems (Control Systems, Robotics)

The theory of stability has its most straightforward and practically impactful applications in the field of engineering control systems analysis and design. The need for a controlled system to return or stay at a desired operating condition be it to a set point, trajectory, or set is primarily one of stability and the whole edifice of modern control theory is based upon foundations of stability analysis.

Nonlinear Control Systems. Stable close loop systems (e.g.

$$ \dot {x} = f(x, u)\quad u = k(x)\tag{8.1} $$

where $ k:\mathbb{R}^n\to \mathbb{R}^m$ is the feedback control law. We will use Lyapunov methods sometimes combined with CLF theory or passivity theory, precisly to analyze the stability of the closed loop system $\dot{x} = f(x, k(x))$ [17]. Recent work on terminal cost and constraint functions that are specifically designed to guarantee closed loop stability for model predictive control (MPC) has been proposed recently[96], [97]. A special but active area of ongoing research is the extension and consistency of stability guarantees to learning based control, where the control law itself is a neural network trained with data[98], [99].

Adaptive Control. Adaptive control systems online adjust their parameters to manage model uncertainty and disturbances, which requires specific tools to evaluate stability. Results of landmark stability imposed Lyapunov based stability conditions for direct and indirect adaptive control schemes. Recent developments comprise composite adaptive laws, L_1 adaptive control, and neural network based adaptive control (e.g. GLuint with stability guarantees).

Robotics. The motion control of robots poses very complicated stability problems: the plant is highly nonlinear (Lagrangian dynamics with multiple degrees of freedom), the tasks are trajectory tracking, not set point regulation, and the environment has contacts/obstacles/uncertainties. Lyapunov techniques for the analysis of stability for robotic systems:(based on error dynamics, passivity based control, contraction).

Recent robotics applications have targeted different aspects of stability: learning based controllers in physical robot systems, safety constrained control using control barrier functions, and the stability of multi robot coordination and formation control.

Control barrier functions (CBFs) have become a widely used tool for safety critical stabilization. $\mathcal{S} = \lbrace x : h(x)ge 0\rbrace$ for the system $\dot{x}=f(x,u)$, then a CBF $h(x)$ satisfies that there exists a class $\mathcal{K}$ function α such that $\dot{h}(x,u) + alpha(h(x))ge 0$, which ensures forward invariance of $\mathcal{S}$. Thus, we combine CLFs (to promote stability) and CBFs (to ensure safety) into a QP framework to provide a unified, computationally tractable solution for safe stabilization of robotic systems.

8.2 Electrical and Power Systems

Power systems are one of the most practically relevant application domains in nonlinear stability theory. Large scale power networks stability, which ensures synchronous generators retain synchronous frequency following a disturbance such as short circuits, line outages or sudden load variation is thus an essential operational requirement for electrical grid reliability.

Power System Stability Classification. Traditionally, power system stability has been classified into rotor angle stability (i.e., the ability to maintain synchronism), voltage stability (the ability to maintain acceptable voltages at all buses in a power system) and frequency stability (the ability to maintain a steady frequency). The physical mechanisms and challenges of nonlinear stability differ from the one category to another.

The swing equation governing the dynamics of a synchronous generator connected to the grid can be written as:

$$M\ddot{\delta} + D\dot{\delta} = P_m   P_e(\delta), \tag{8.2} $$

Note that δ is the rotor angle; M, inertia constant; D, damping coefficient; P_m, mechanical power input; and: \begin{equation} P_{e}(\delta) = \frac{EV}{X}\sin\delta end{equation}. This is a form of pendulum nonlinear ODE, and the stability analysis (single machine infinite bus systems) can be done using an energy like function that can be reached directly by Lyapunov methods:

Your first exact data (to October 2023):$$V(\delta, \dot{\delta}) = \frac{1}{2}M\dot{\delta}^{2} + M\int_{\delta_{s}}^{δ}(P_{e(θ)} − P_{m}), dθ, (8.3)$$

Called the transient energy function (TEF) The TEF based stability analysis directly provides estimates the critical clearing time, which is defined as the maximum fault duration beyond which the system will not be able to recover stability, and serves as a foundation for extended equal area criterion (EEAC).

The fundamental characteristic of the modern power system stability problems is the accessibility of renewable energy sources like wind energy and solar energy through fast, highly nonlinear dynamics through their interfacing power electronic converters. This is impelling researchers to explore different theoretical frameworks for stability analysing inverter dominated grids, which does not follow classical synchronous machine paradigm. Nonlinear stability theory is used to evaluate virtual oscillator control and droop control strategies, with Lyapunov functions being constructed based on port Hamiltonian system formulations.

Voltage stability refers to the behaviour of power systems in response to disturbances when the load demand approaches or exceeds the maximum load carrying capability [52]. The closeness of the operating point to its nearest bifurcation the voltage stability margin is an important operational indicator, and involves performing sensitivity analysis as well as solving nonlinear eigenvalue problems.

8.3 Biological and Ecological Models

Nonlinear stability theory has a long and successful history of application to biological and ecological systems as well as an active presence in modern literature.

Population Dynamics. The Lotka–Volterra predator prey system:

So we have the equations $\dot{N}1 = r_1 N_1   a{12} N_1 N_2, \quad \dot{N}2 = r_2 N_2 + a{21} (8.4)$$

has an equilibrium at $(N_1^, N_2^) = (r_2/a_{21}, r_1/a_{12})$ that is a center: Lyapunov stable but not asymptotically stable. The Lyapunov function for this system, $V = a_{21}N_1−r_2ln

N_1+a_{12}N_2−r_1ln N_2$, ensures the stability but stresses the conservative (non dissipative) nature of classical Lotka–Volterra model. Numerous models with more realism that consider the impacts of density dependence, functional responses (Holling types I, II, III), and environmental stochasticity have been studied for asymptotic and global stability.

Epidemiological Models. The stability of the Compartmental Epidemiological Models SIR, SEIR, SEIRS & their generalisation are Analyzing mathematical model from which epidemic dynamics can be understood and Control interventions can evaluate against such as a viruses. For these models, the disease free equilibrium (DFE) is asymptotically stable when the basic reproduction number $\mathcal{R}_0 1$). For $\mathcal{R}_0 > 1$ the endemic equilibrium is generally globally asymptotically stable by Lyapunov methods with Volterra type functions for standard SEIR models.

The COVID 19 pandemic triggered a flood of stability analyses focusing on the reality based generalizations of extended compartmental models such as those with age structure, spatial heterogeneity, vaccination and behavioral responses. State variables easily surpass ten in such models, requiring state space global stability analysis via unique Lyapunov function constructions and graph theoretic methodologies[100].

Neural Network Dynamics. From both a biological and computational standpoint, the stability of Hopfield neural networks (HNNs) and recurrent neural networks (RNNs) are questioned. The energy function is $V(x) = \frac{1}{2}x^T W x + \mathbf{b}^T x$ which is a Lyapunov function, and decreases along trajectories for Hopfield networks with symmetric weights $W = W^T$, thus ensuring convergence to an equilibrium. LMI based Lyapunov methods have been used to analyze asymptotic stability of general RNNs[101] and the dynamics (gradient flows) for deep neural networks during training is an emerging active research field[102].

Biochemical Reaction Networks. Stability analysis of chemical reaction networks (CRNs), systems of ODEs describing the concentrations of reacting chemical species, is motivated by applications in systems biology and synthetic biology. The Deficiency Zero Theorem is a structural theory providing conditions on network topology (stoichiometry and kinetics) under which the system possesses a unique asymptotically stable equilibrium in each compatibility class for all parameter values, a surprising classical result of structural stability[103]. Recent extensions include stability analysis of enzymatic networks, dynamics of MAPK cascades and gene regulatory networks.

8.4 Mechanical Systems

Lagrangian or Hamiltonian equations of motion describe mechanical systems and constitute a familiar but complex class of nonlinear differential equations, with stability theory already possessing both extensive practical applications and several deep theoretical results.

Hamiltonian Systems. The Hamiltonian equations of motion for a conservative mechanical system with n degrees of freedom are:

where $$\dot{q} = \frac{\partial H}{\partial p}, \quad dot{p} = \frac{\partial H}{\partial q},eqref(8.5)$$

where $q \in \mathbb{R}^n$ the generalized coordinates, $p \in \mathbb{R}^n$ the conjugate momenta and $H(q,p)$ is total energy (the Hamiltonian). In such systems, $dot{H} = 0$ on trajectories: energy is conserved. The equilibria are critical points of the potential energy $U(q)$ (with Hamiltonian as $H = T + U$, for kinetic energy T), and stability is a second order property on U: $(q^, 0)$ is a Lyapunov stable equilibrium of (8.5) if and only if $q^$ is a strict local minimum of U by the Lagrange–Dirichlet theorem. Nevertheless, due to energy conservation asymptotic stability requires dissipative forces the equilibrium is not asymptotically stable.

Dirac structure and port Hamiltonian systems generalizes Hamiltonian mechanics to include dissipative effects or input output behaviour, e.g., appeared in stability analysis of mechanical systems whit friction (blood flow) fluid structure interaction and multibody dynamics.

Vibration Control and Active Damping. Mechanical vibrations be it of structures, rotating machinery or vehicles can say to have always been more a classical engineering concern with immediate implications for the aerospace, civil and automotive systems. Active vibration control employs state estimators and actuators to inject damping forces, and the closed loop system behavior is examined with Lyapunov tools adapted to second order mechanical systems. Recent activities involve the stability analysis of piezoelectric smart structures, magneto rheological damper systems and energy harvesting vibration systems.

9. Recent Advances and Literature Review

9.1 Major Contributions To Research (Past 10 Years)

Recent Advances in Nonlinear Stability Analysis [2015–2026] The discussion of key contributions is organized by methodological theme.

Advances in Lyapunov Theory. Major improvements of converse Lyapunov theorems for non uniform exponential stability were obtained even when compactly supported smooth Liapunov functions are produced under almost no graining assumptions with respect to the Dynamics. These results clear the ambiguity between stability and existence of smooth Lyapunov certificates, with implications for designing stability verifying optimization algorithms. Two decades of extensions and applications of comprehensive surveys of input to state stability and its relations to Lyapunov theory are assembled in a single framework.

The finite dimensional ISS theory has been extended to infinite dimensional systems by providing foundational ISS Lyapunov characterizations for PDEs in Banach spaces[104]. We focus on wave equations, heat equations and more general evolution equations, providing Lyapunov functional conditions for ISS that are strikingly analogous to the finite dimensional theory[13]. This contribution is directly linked to the linear matrix inequalities for stability analysis of DPM systems.

Computational Stability Verification. SOS based approaches to gain estimates of exponential decay rates and regions of attraction for polynomial non linear systems via hierarchies of SDP relaxations generate optimally tight Lyapunov based bounds[105]. In contrast to the polynomial system restriction, contraction based frameworks calculated from matrix measure analysis can achieve global validity of a verified Lyapunov function for incrementally stable systems[106].

Propose a systematic approach for learning neural Lyapunov functions with formal verification that combines neural network synthesis and SMT based verification[107]. These methods have been shown on systems with dimension up to n=6 and provided methodological frameworks that were followed and expanded by many works afterwards. CEGIS based Lyapunov function synthesis verification loops come with convergence guarantees. Neural Lyapunov methods have also been generalized to hybrid systems[108], where stability verification is complicated by discrete state transitions.

Fractional Order Systems. Stability is one of the most important property, which has been researched for past decade for mutual systems. Background Comprehensive Lyapunov function based stability analysis frameworks for Caputo fractional order systems establish generalized conditions for asymptotic and exponential stability over the integer order theory[109]. Nonlinear Dynamics Global stability of fractional order Lotka–Volterra systems based on a class of Lyapunov function: An application on biological model[110]. The stability analysis of delayed fractional order neural networks has also been studied applying Lyapunov–Krasovskii functionals to the control strategy in the general framework of fractional differential systems.

Time Delay Systems. Contributions to stability analysis of nonlinear systems with time varying delays have been made by using augmented variable techniques and reciprocal convexity lemmas to construct improved Lyapunov–Krasovskii functionals[111]. LMI based stability criteria tend to ease conservatism relative to previous approaches. Substantial extension of ISS theory has appeared for nonlinear time delay systems–characterizations of ISS in terms of Razumikhin type conditions and Lyapunov–Krasovskii functionals with links to finite dimensional ISS theory[112].

Stochastic Systems. Abstract Comprehensive ISS (input to state stability) frameworks for stochastic nonlinear systems capture the concept of stochastic ISS using Lyapunov functions that satisfy point dissipation conditions. These are all unified treatments of ISS for Itô SDEs that mimic the deterministic theory. Also, the global stability of stochastic nonlinear systems under lessGrowth conditions were investigated through new construction of Lyapunov functions in the framework of Stochastic Systems.

Hybrid and Switched Systems. The past ten years have seen significant extensions of foundational results on stability of switched systems via common and multiple Lyapunov functions. Frameworks for stability of hybrid dynamical systems systems with both continuous time dynamics and discrete jumps use adapted Lyapunov functions and invariance principles to the hybrid setting. Extensive subsequent research has largely built on foundational textbook treatments of hybrid dynamical systems stability.

9.2 Comparative Analysis of Methods

Specifically, a systematic comparison of the main methods for nonlinear stability analysis highlights distinctive trade offs in analytical power, computational tractability, domain of applicability and strength of conclusions. Summary of the major aspects of comparison in Table 9.1 (described textually below).

Lyapunov Direct Method (Analytical): Classical analytical Lyapunov approach has solvability guarantees (sufficient conditions that become necessary due to converse theorems) and operates with a much broader class of systems arbitrary ODE with locally Lipschitz right hand side. In contrast, it requires the analyst to find or build a Lyapunov function–which is as much an art as a science. This provides global or regional stability guarantees depending on the chosen Lyapunov function, but this doesn ˆ t automatically give you the largest possible region of attraction. Its computational complexity is essentially driven by the analyst's creativity rather than a systematic procedure.

SOS Programming: SOS approaches, through SDP, systematically search for systems in a polynomial class and capture stability certificates in an optimally verifiable and computationally tractable form. They are also complete (with respect to a given degree of polynomial Lyapunov functions) in that if a polynomial Lyapunov function exists, the SOS method is guaranteed to find it. They are limited to polynomial systems, and the computational complexity is prohibitive as it scales rapidly with system dimension and polynomial degree approximately $O(n^{2d})$ SDP variables for a degree d, limiting practice applicability to about 10–20 dimensions. However, the estimation of region of attraction heavily depends on the Lyapunov function degree and domain parameterization.

Neural Network Lyapunov Methods: These approaches use neural networks, sacrificing formal completeness (i.e. no convergence guarantees generally) for scalability applications to systems of dimension above 10 (well beyond the reach of SOS methods). It is not confined to polynomial systems and can be applied to the systems with transcendental nonlinearities. Nonetheless the formal verification step is needed to certify the learnt Lyapunov function incurs difficult computational problems that severely restricts verified applicability to moderate dimensions and disarmingly simple domains[113], [114].

Linearization Based Methods: Linearization offers local stability analysis at almost no cost (only an eigenvalue computation must be performed), making this type the preferred initial method to begin the analysis. On the other hand, it reports local results without any data on the size of basin of attraction and is inconclusive at degenerate (center) case. When analyzing global stability or systems operating far from equilibrium, linearization is fundamentally inadequate[115], [116].

Facilitated by the ability to analyze global exponential stability and interconnected/time varying systems. They do not need a specific Lyapunov function but instead need to check a matrix measure condition which might be verified via LMI methods. The main drawback is that not all asymptotically stable systems are contracting; contraction is a stricter characteristic which does not hold for multi equilibrium or limit cycle systems[117], [118].

Comparison Principles: Comparison principles are typically best for large systems that are connected and there is a way to take advantage of the coupling structure. M matrix conditions are a classical set of elegant sufficient stability conditions for network dynamics, with only partial competitor interactions considered. On the other hand, comparison principles enforce a degree of conservativeness by enclosing the full nonlinear interaction with significantly simpler (linear) comparison functions and this conservatism can lead to inapplicability of conditions even for strongly coupled systems[119], [120].

9.3 New Trends in Stability Theory:

A number of methodological trends are currently driving the development of nonlinear stability theory, shaped by both the demands of new applications (safety critical systems, distributed network control, biological systems) and technological opportunities (machine learning / AI tools for model & parameter reduction, formal verification tools enabling higher order estimates [14], high performance computing facilitating abstract stability computations).

Data Driven and Learning Based Stability Analysis. The simulation data or experimental measurements that can provide much more information at a very lower computational cost, together with the fast scalable and robust methods introduced by machine learning recently have pushed people to go for data driven approaches for stability analytical tasks. Abstract This paper elaborates on three different frameworks for learning stability certifi cates from data: Neural Lyapunov functions, Gaussian process based stability certification and Koopman operator based stability analyses. In particular, the Koopman operator framework lifts nonlinear dynamics into a (in principle infinite dimensional) linear space where linear stability analysis applies, and it bridges data driven learning with classical Lyapunov theory.

Stability of Learning Systems. Now viewed as an "up and coming" research frontier, machine learning system stability has increasingly attracted the attention of researchers in areas such as the stability of optimization dynamics (gradient flow)[121], recurrence network training methods[122], and RL policies[123]. Lyapunov methods have been applied to show the stability of gradient flow dynamics for deep neural network training[124], and connections with neural tangent kernels make this a more active area of research[125]. The stability guarantees provided by CLF and CBF theory especially with regards to the safety of RL policies have driven their incorporation into new RL frameworks[126].

Cyber physical & Communication Constraints: Stability The coupling of physical systems with digital controllers through networked communication has moved forward stability problems associated with packet dropouts, communication delays and quantization and sampling effects[127]. The existing event triggered and self triggered control frameworks assume stability in the presence of communication constraints using Lyapunov methods that take into direct account the inter event dynamics[128]. The main analytical tools of the field are thanks to the ISS and input output stability frameworks[129].

Mean Field and Network Dynamics. Applications including mean field game theory, coupled oscillator networks and multi agent system dynamics are novel examples where nonlinear stability theory is used[130].

Graph theoretic Lyapunov methods have been used to analyze the stability of networks of Kuramoto oscillators[131], and Wasserstein metric Lyapunov functions have been employed to study the stability of mean field limit dynamics for large networks of interacting particles[132]. Such applications will therefore request stability frameworks suited for infinite dimensional or graph structured state spaces.

Challenges and Open Problems

10.1 Limitations of Existing Methods

The theoretical and computational advancements outlined previously notwithstanding, many of the existing methods for stability analysis are limited. These limitations characterize the frontier of the research area and motivate some open problems described below.

The second and this more serious major limitation is ineffective or generally useless methodology base lyapunov function construction methods for nonlinear systems[133], [134]. Although converse Lyapunov theorems promise that smooth Lyapunov functions always exist whenever asymptotic stability does, all constructive methods emerged with broad but still very limited domains in SOS programming (Henrion & Lasserre 2006), variable gradient methods (Molchanov & Sonin 2018), and neural network synthesis[135]. The design of Lyapunov functions for non polynomial, high dimensional or nonsmooth systems is mostly open and the stability certificate is limited to system classes that are tractable by existing methods[134], [136].

The second key drawback is the conservativeness of some current stability conditions. The gap between the sufficient conditions and true stability boundary is typically large, and most Lyapunov based analysis/symmetry/comparison principles/LMI formulations provide only sufficient but not necessary ones. This conservative control shows up practically as either very small bounds on the allowable perturbations of a plant, couplings these ranges with coupling gains or operating points too small for stability analysis to assist controller design. The Challenge of not Being Too Conservative and Yet Computationally Tractable

The third limitation is in the scalability to higher dimensional systems[113]. This can be very limiting, as shown in Section 7.4: the computational effort to prove stability of high dimensional systems is overwhelming; such systems cannot be verified by formal methods from a practical perspective[113]. The systems that emerge from spatial discretisation of PDEs, multi agent networks and a variety of high dimensional biological models regularly exceed the dimensionality thresholds beyond which rigorous methods become tractable.

The fourth limitation is the structural model uncertainty. The stability analysis frameworks usually assume that system equations are known with moderate accuracy. In practice, system models are necessarily approximate and conclusions on stability will not transfer from a nominal model to the realization of the true system. The small perturbation uncertainty can be handled by ISS and robustness frameworks; however, dealing systematically with large scale structural model uncertainty or discrepancy between the model and system remains very challenging.

10.2 High Dimensional Nonlinear Systems

The stability analysis of high dimensional nonlinear systems state dimensions in the order of 50–1000 or even more is also different qualitatively from the moderate dimensional case. Such systems occur in network dynamics, discretized PDEs, systems biology and machine learning[137], [138], their analysis demands approaches leveraging structural properties (sparsity, symmetry, hierarchical decomposition) to avoid the curse of dimensionality[139].

Decomposition and Aggregation Methods. In particular, hierarchical stability analysis breaks down high dimensional systems into lower dimensional subsystems which can be treated independently and combines the resulting local stability results to provide global guarantees[140]. With the vector Lyapunov function approach, we can assign each subsystem a scalar Lyapunov function and then create a vector comparison system where the inter subsystem coupling terms enter into it[141]. If this comparative system is stable, global stability follows a trait expressible through M matrix theory[113]. The main difficulty with decomposition methods is to control the conservatism that arises from decoupled analysis, which may not identify favourable interactions between subsystems.

Exploiting Sparsity and Network Structure. Stability analysis can exploit the graph structure to obtain conditions that scale linearly or polynomially in network size for sparse coupling graphs [15]. Under ISS small gain conditions, interconnected systems can be verified solely in terms of the gain matrices of the individual subsystems (with the network structure encoded in the interconnection Jacobian)[142]. For systems on graphs, spectral conditions (involving the graph Laplacian eigenvalues) give scalability independent of network size for certain coupling patterns[143].

Reduced Order Modeling. Instead of going through this whole process, an alternative route to manageable analysis of a higher dimensional system is stability analysis of reduced order models achieved via POD, DMD, the Koopman operator approach or balanced truncation. The main issue is to show that stability of the reduced order model guarantees stability of the full order system a question that involves an careful analysis of the approximation error (which may affect stability conclusions).

Incremental Stability and Contractivity in Really High Dimensions. Contraction analysis has the advantage of avoiding a curse of dimensionality, as it only needs to ensure that for all x, $\mu(Df(x))\leq \lambda 0$, $\lim_{x_0 \to 0} P(\sup_{t \geq 0} |x(t)| > \varepsilon) = 0$.

- Almost sure asymptotic stability: $$P(\lim_{t to infty} x(t) = 0)=1, |x_0| sup$$ small.

- Mean square asymptotic stability: $ \lim\limits_{t \to +\infty} \mathbb{E}[|x(t)|^2] = 0 $.

- Mean square Exponential Stability: $\mathbb{E}[|x(t)|^2] \leq ce^{ \lambda t} \mathbb{E}[|x_0|^2].$

The two concepts are closely related but not identical; loosely speaking, mean square stability implies stability in probability for any 0<ε<1 and exponential mean square stability is also equivalent to almost sure exponential stability (following the Borel–Cantelli lemma), but converse does not hold [36].

Stochastic Lyapunov Theory. The Lyapunov method is generalised to SDEs in a natural way via the infinitesimal generator. For the Itô SDE (3.7), we have $\mathcal{L}$ acting on twice differentiable functions V:

The 2nd order backward stochastic differential operator \mathcal{L} is given by: \begin{align*} \mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2}\text{tr}\left\sigma(x)^T \frac{\partial^2 V}{\partial x^2} σ\right]. \tag{10.1}

The Lyapunov stability theorem generalized for stochastic systemsask to havefor a positive definite function $\textbf{V}$, $\mathcal{L}V(x) \leq W(x)$ with; The term $\mathcal{L}V$, involving the diffusion, gives rise to an interaction between stabilizing forces in the drift f and destabilizing diffusion effects and vice versa, which has no deterministic analogue hence the idea of noise induced stability.

Open problems on Stochastic stability Currently, there is several key open problems in stochastic nonlinear stability that drive research:

(i) Converse Lyapunov theorems for random systems: although converse results have been established for stability in probability and almost sure stability since the 1980s, complete converse theorems for ISS of SDEs with general diffusion coefficients were only recently proved under special cases. Full generalization remains open[144], [145].

(ii) ISS and robustness in high dimensional SDEs: Finally, the extension of ISS theory to high dimensional stochastic systems including network coupled SDEs and stochastic PDEs (SPDEs) appears to be a hot research area with many open problems. More broadly, how noise interacts with network coupling to determine stability is poorly understood.

(iii) Data enhanced stochastic stability analysis: This is a research area that has important applications in probabilistic modeling of biological systems as well financial systems, but remains deeply challenging since one can only observe trajectory data (rather than the SDE drift f or diffusion σ leading to those trajectories). Neural SDE frameworks offer a natural way of initializing the processes, however formal stability guarantees from data driven stochastic models are still largely lacking.

(iv) Stability of McKean–Vlasov SDEs. Systems of type $dx_t = f(x_t, \mu_t);dt + \sigma(x_t,\mu) dW_t,$ μ is the law of random variable x, appear in mean field game and many agent system theory. Tools from optimal transport and Wasserstein geometry are needed for their stability analysis, which is essentially an open area of research.

(v) Stability in the presence of Lévy noise: A large class of practical noise processes with examples including financial assets dynamics, earthquake vibrations, and computer networks traffic fluctuations are modeled by Lévy processes rather than Brownian motion. SDEs which are driven by Lévy noise are not very advanced compared to the Itô theory as most of the questions related to converse Lyapunov theorem (ISS, global and/or uniform exponential stability) remains unsolved.

These challenges are among the most technologically challenging and practically relevant problems in modern nonlinear stability theory, and addressing them demands the synthesis of methods from stochastic analysis, functional analysis, control theory, and numerical computation.

11. Conclusion

In fact, this survey has described the theory, methods and applications of asymptotic and exponential stability for nonlinear differential equations in detail and with good analytical rigor. Starting from the abstract mathematical principles concepts of equilibrium, comparison functions, and definitions of Lyapunov stability, asymptotic stability, and exponential stability this review developed step by step the main analysis techniques: Lyapunov's direct method, LaSalle invariance principle, linearization methods, construction procedures for Lyapunov functions; fixed point theorems; comparison principles. Stability Analysis Computational Dimension: This aspect of stability analysis has been considered in surveys of numerical Lyapunov function computation, simulation techniques, software tools and fundamental challenges with high dimensional systems and non smooth dynamics.

As highlighted in the applications sections, nonlinear stability theory provides an broadly applicable and practically relevant framework extending to control and robotics, electrical power systems, biological and ecological models, mechanical systems. Most significant methodological breakthroughs of the recent decade, including neural Lyapunov function synthesis, SOS based regional stability estimation, contraction based global stability and ISS extensions to infinite dimensional systems have been featured in the section on recent advances along with identification of methodological trade offs that can act as guiding principles for approach selection for specific problem classes.

As we saw in the challenges and open problems section, this leaves us with three frontiers that will take research efforts in both analytical and computational directions: limitations of existing analytical and computational methods, scalability issues due to high dimensionality of many systems, and large open problems for stochastic nonlinear stability. Together, these frontiers outline the research agenda for the next decade; they indicate where inter disciplinary research in control theory, optimization, formal verification, machine learning and stochastic analysis is required.

This synthesis provides a number of broad observations. First, the core observation of Lyapunov that stability can be expressed in terms of scalar energy like functions without waveforms continues to be as fertile and algorithmically generative today as it was thirty years ago[146], [147]. The extension of this insight to neural network synthesis, SOS programming, and contraction metrics is a modern day embodiment of a principle over one hundred years old that continues to manifest in new forms[69], [148]. Second, the tradeoff between analytical sufficiency and computational tractabilityis the axis around which this field organizes itself: theoretically complete methods (e.g.,converse Lyapunov theorems force us to a need for witnesses; like SOS approaches) are alsocomputationally intractable in general, while work horse techniques that are computably efficient(SOS, neural networks) are theoretically incomplete (only apply to certain system classes orwithout convergence guarantees)[113], [149]. The challenge that dominates the field is breaking this tension how to design effective stability analysis methods that are both theoretically grounded and computationally scalable.

Third, the treatment of uncertainty, noise and model error in a stability analyse is an area that remain to be fully addressed[150]. The ISS framework offers a powerful formalism for deterministic robust stability with respect to uncertainty[151], [152]., but its extension in the presence of stochastic uncertainty, structural model error and distributional shift provide open challenges that are only partially solved at this point[153], [154]. Visioning multiple stability frameworks robust to all practically relevant sources of uncertainty avoiding over conservatism is an important goal for fundamental and applied research.

References

[1] C. W. Gear, J. M. Hyman, P. G. Kevrekidid, I. G. Kevrekidis, O. Runborg, and C. Theodoropoulos, “Equation Free, Coarse Grained Multiscale Computation: Enabling Mocroscopic Simulators to Perform System Level Analysis,” *Communications in Mathematical Sciences* , vol. 1, no. 4, pp. 715–762, Jan. 2003, doi: 10.4310/cms.2003.v1.n4.a5.

[2] I. Markovsky and F. Dörfler, “Behavioral systems theory in data driven analysis, signal processing, and control,” *Annual Reviews in Control* , vol. 52, pp. 42–64, Jan. 2021, doi: 10.1016/j.arcontrol.2021.09.005.

[3] T. Stamov, “Neural Networks in Engineering Design: Robust Practical Stability Analysis,” *Cybernetics and Information Technologies* , vol. 21, no. 4, pp. 3–14, Dec. 2021, doi: 10.2478/cait 2021 0039.

[4] B. Carpentieri, *Advances in Dynamical Systems Theory, Models, Algorithms and Applications* . IntechOpen, 2021. doi: 10.5772/intechopen.92486.

[5] V. Komorin, “ASSESSMENT OF THE BLACK SEA SHELF ECOSYSTEM SUSTAINABILITY WITH MATHEMATICAL SIMULATION METHOD,” *Geographia Technica* , vol. 16, no. 2, pp. 19–28, Jun. 2021, doi: 10.21163/gt_2021.162.02.

[6] X. Guo *et al.* , “ControlAgent: Automating Control System Design via Novel Integration of LLM Agents and Domain Expertise,” *arXiv (Cornell University)* , Oct. 2024, doi: 10.48550/arxiv.2410.19811.

[7] T. Ren, Z. Ren, N. Li, and B. Dai, “Stochastic Nonlinear Control via Finite dimensional Spectral Dynamic Embedding,” *arXiv (Cornell University)* , Apr. 2023, doi: 10.48550/arxiv.2304.03907.

[8] L. Celentano, “A fast design technique for robust industrial controllers,” *Journal of the Franklin Institute* , vol. 360, no. 8, pp. 5689–5727, Apr. 2023, doi: 10.1016/j.jfranklin.2023.03.033.

[9] S. Pathak and V. R. Kota, “Mathematical Analysis of an Infectious Disease Model: Stability, Persistence, and Equilibrium Analysis in a Non Linear Epidemiological System,” *Journal of Nonlinear Mathematical Physics* , vol. 32, no. 1, Oct. 2025, doi: 10.1007/s44198 025 00333 1.

[10] H. Chen, B. Haus, and P. Mercorelli, “Extension of SEIR Compartmental Models for Constructive Lyapunov Control of COVID 19 and Analysis in Terms of Practical Stability,” *Mathematics* , vol. 9, no. 17, pp. 2076–2076, Aug. 2021, doi: 10.3390/math9172076.

[11] J. Yang, C. K. Tse, and M. Huang, “Complexity of Grid Connected Power Electronics Converters and Its Impacts on Modern Power Grids,” *International Journal of Bifurcation and Chaos* , vol. 35, no. 14, Oct. 2025, doi: 10.1142/s0218127425400061.

[12] G. F. Silva, A. Donaire, R. H. Middleton, A. McFadyen, and J. Ford, “Scalable Input to State Stability of Nonlinear Interconnected Systems,” *IEEE Transactions on Automatic Control* , vol. 70, no. 3, pp. 1824–1834, Sep. 2024, doi: 10.1109/tac.2024.3468069.

[13] A. Mironchenko and C. Prieur, “Input to State Stability of Infinite Dimensional Systems: Foundations and Present Day Developments,” in *Elsevier eBooks* , Elsevier BV, 2025. doi: 10.1016/b978 0 443 14081 5.00066 0.

[14] “Prelims.” Oct. 27, 2025. doi: 10.1108/978 1 80592 517 020251014.

[15] Y. C. Aybas and M. O. Jackson, “Social Learning with Endogenous Information and the Countervailing Effects of Homophily,” *ArXiv.org* , Jan. 2026, Accessed: Feb. 2026. [Online]. Available: http://arxiv.org/abs/2602.00934

[16] A. R. Ali, H. O. Roshid, S. Islam, and A. Khatun, “Analyzing bifurcation, stability, and wave solutions in nonlinear telecommunications models using transmission lines, Hamiltonian and Jacobian techniques,” *Scientific Reports* , vol. 14, no. 1, Jul. 2024, doi: 10.1038/s41598 024 64788 w.

[17] J. P. Zubelli, J. Loria, and V. Albani, “On the Estimation of the Time Dependent Transmission Rate in Epidemiological Models,” *arXiv (Cornell University)* , Dec. 2024, doi: 10.48550/arxiv.2412.19930.

[18] Y. Cai and Y. J. Duan, “Achieving Universal Approximation and Universal Interpolation via Nonlinearity of Control Families,” *arXiv (Cornell University)* , Oct. 2025, doi: 10.48550/arxiv.2510.03676.

[19] R. Haider, X. Li, W. Shi, Z. Lin, Q. Xiao, and H. Zhao, “Review of Computational Fluid Dynamics in the Design of Floating Offshore Wind Turbines,” *Energies* , vol. 17, no. 17, pp. 4269–4269, Aug. 2024, doi: 10.3390/en17174269.

[20] R. Zen and M. 10, “The Emergence of Universal Attractors in Non Autonomous Dynamical Systems,” *Zenodo (CERN European Organization for Nuclear Research)* , Dec. 2025, doi: 10.5281/zenodo.17838542.

[21] A. H. Alshammari, N. Aswad, and M. B. Alshammari, “Stability of the zero solution of fuzzy delay differential equations via Lyapunov functions,” *Automatika* , vol. 67, no. 1, pp. 222–236, Feb. 2026, doi: 10.1080/00051144.2026.2620854.

[22] S. D. A. PAULO, “Stability and Invariant Manifolds for Stochastic Evolution Equations,” *Zenodo (CERN European Organization for Nuclear Research)* , Nov. 2025, doi: 10.5281/zenodo.17683344.

[23] T. T. Nguyên *et al.* , “A Survey of Machine Unlearning,” *ACM Transactions on Intelligent Systems and Technology* , vol. 16, no. 5, pp. 1–46, Jul. 2025, doi: 10.1145/3749987.

[24] H. T. N., H. M.A., and M. Hammi, “On the global uniform stability analysis of non autonomous dynamical systems: A survey,” *Mathematica Moravica* , vol. 26, no. 2, pp. 1–48, Jan. 2022, doi: 10.5937/matmor2202001t.

[25] D. Lee and D. W. Kim, “On the Local Quadratic Stability of T–S Fuzzy Systems in the Vicinity of the Origin,” *Fuzzy Sets and Systems* , vol. 497, pp. 109113–109113, Aug. 2024, doi: 10.1016/j.fss.2024.109113.

[26] R. Vrábeľ, “Criterion for robustness of global asymptotic stability to external perturbations of linear time varying systems,” *International Journal of General Systems* , vol. 50, no. 2, pp. 211–222, Jan. 2021, doi: 10.1080/03081079.2020.1870223.

[27] C. Guiver and H. Logemann, “The exponential input to state stability property: characterisations and feedback connections,” *Mathematics of Control Signals and Systems* , vol. 35, no. 2, pp. 375–398, Jan. 2023, doi: 10.1007/s00498 023 00344 7.

[28] Q. Wang, “Control Design for Transient Performance,” *IEEE/CAA Journal of Automatica Sinica* , vol. 10, no. 1, pp. 3–7, Jan. 2023, doi: 10.1109/jas.2023.123006.

[29] A. N. Nekhoroshikh, “Implicit Lyapunov methods for analysis and synthesis of superexponentially stable systems,” *HAL (Le Centre pour la Communication Scientifique Directe)* , Oct. 2022, Accessed: Oct. 2025. [Online]. Available: https://hal.inria.fr/tel 03991799

[30] A. Chaillet, I. Karafyllis, P. Pepe, and Y. Wang, “The ISS framework for time delay systems: a survey,” *Mathematics of Control Signals and Systems* , vol. 35, no. 2, pp. 237–306, Jan. 2023, doi: 10.1007/s00498 023 00341 w.

[31] W. Jongeneel and E. Moulay, *Topological Obstructions to Stability and Stabilization* . Springer International Publishing, 2023. doi: 10.1007/978 3 031 30133 9.

[32] A. Fontan, *Collective decision making on networked systems in presence of antagonistic interactions* . Linköping University Electronic Press, 2021. doi: 10.3384/diss.diva 178301.

[33] P. Mestres, Y. Chen, E. Dall’Anese, and J. Cortés, “Control Barrier Function Based Safety Filters: Characterization of Undesired Equilibria, Unbounded Trajectories, and Limit Cycles,” *arXiv (Cornell University)* , Jan. 2025, doi: 10.48550/arxiv.2501.09289.

[34] S. Kaakaï, A. Matoussi, and A. Tamtalini, “Estimation of Systemic Shortfall Risk Measure Using Stochastic Algorithms,” *SIAM Journal on Financial Mathematics* , vol. 15, no. 3, pp. 700–733, Aug. 2024, doi: 10.1137/22m1539344.

[35] C. Schlosser, “Converging Approximations of Attractors via Almost Lyapunov Functions and Semidefinite Programming,” *IEEE Control Systems Letters* , vol. 6, pp. 2912–2917, Jan. 2022, doi: 10.1109/lcsys.2022.3180110.

[36] A. Mattioni, “Modelling and stability analysis of flexible robots : a distributed parameter port Hamiltonian approach,” *HAL (Le Centre pour la Communication Scientifique Directe)* , Apr. 2021, Accessed: Jul. 2025. [Online]. Available: https://theses.hal.science/tel 03454705

[37] W. Tan and A. Packard, “Stability Region Analysis Using Polynomial and Composite Polynomial Lyapunov Functions and Sum of Squares Programming,” *IEEE Transactions on Automatic Control* , vol. 53, no. 2, pp. 565–571, Mar. 2008, doi: 10.1109/tac.2007.914221.

[38] A. L. J. Bertolin, R. C. L. F. Oliveira, G. Valmórbida, and P. L. D. Peres, “An LMI Approach for Stability Analysis and Output Feedback Stabilization of Discrete Time Lur’e Systems Using Zames Falb Multipliers,” *IEEE Control Systems Letters* , vol. 6, pp. 710–715, Jun. 2021, doi: 10.1109/lcsys.2021.3086427.

[39] S. Jalbi and S. Bhattacharya, “Closed form solution for the first natural frequency of offshore wind turbine jackets supported on multiple foundations incorporating soil structure interaction,” *Soil Dynamics and Earthquake Engineering* , vol. 113, pp. 593–613, Jul. 2018, doi: 10.1016/j.soildyn.2018.06.011.

[40] T. C. Lee, Y. Tan, and I. Mareels, “Limiting Behavior of Hybrid Time Varying Systems,” *IEEE Transactions on Automatic Control* , vol. 67, no. 11, pp. 5777–5792, Oct. 2021, doi: 10.1109/tac.2021.3122376.

[41] I. Karafyllis and A. Chaillet, “Lyapunov conditions for uniform asymptotic output stability and a relaxation of Barbălat’s lemma,” *Automatica* , vol. 132, pp. 109792–109792, Jul. 2021, doi: 10.1016/j.automatica.2021.109792.

[42] M. N. Dao, H. Saoud, and M. Théra, “Locating Theorems of Differential Inclusions Governed by Maximally Monotone Operators *,” *HAL (Le Centre pour la Communication Scientifique Directe)* , Oct. 2022, Accessed: Oct. 2025. [Online]. Available: https://hal.science/hal 03810134

[43] A. Sel, M. Koruturk, and E. Sayar, “Estimation of Regions of Attraction for Nonlinear Systems via Coordinate Transformed TS Models and Piecewise Quadratic Lyapunov Functions,” *arXiv (Cornell University)* , Jul. 2025, doi: 10.48550/arxiv.2507.12718.

[44] I. B. Furtat and П. А. Гущин, “Stability/Instability Study and Control of Autonomous Dynamical Systems: Divergence Method,” *IEEE Access* , vol. 9, pp. 23764–23771, Jan. 2021, doi: 10.1109/access.2021.3056942.

[45] B. Haasdonk, B. Hamzi, G. Santin, and D. Wittwar, "Kernel methods for center manifold approximation and a weak data based version of the Center Manifold Theorem," *Physica D Nonlinear Phenomena* , vol. 427, pp. 133007–133007, Aug. 2021, doi: 10.1016/j.physd.2021.133007.

[46] S. J. Sherwin, D. Moxey, J. Peiró, P. Vincent, and C. Schwab, *Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2018* . Springer Nature, 2020. doi: 10.1007/978 3 030 39647 3.

[47] H. Hjalmarsson, "From experiment design to closed loop control," *Automatica* , vol. 41, no. 3, pp. 393–438, Jan. 2005, doi: 10.1016/j.automatica.2004.11.021.

[48] C. Guiver and H. Logemann, "A circle criterion for strong integral input to state stability," *Automatica* , vol. 111, pp. 108641–108641, Oct. 2019, doi: 10.1016/j.automatica.2019.108641.

[49] W. Jongeneel, "Asymptotic stability equals exponential stability while you twist your eyes," *arXiv (Cornell University)* , Nov. 2024, doi: 10.48550/arxiv.2411.03277.

[50] M. Tripathi, L. Moysis, M. K. Gupta, G. F. Fragulis, and C. Volos, "Observer Design for Nonlinear Descriptor Systems: A Survey on System Nonlinearities," *Circuits Systems and Signal Processing* , vol. 43, no. 5, pp. 2853–2872, Feb. 2024, doi: 10.1007/s00034 024 02617 1.

[51] A. Polyakov, D. Efimov, and X. Ping, "Energetically Optimal Finite Time Stabilization of Generalized Homogeneous Linear Systems," *IEEE Transactions on Automatic Control* , vol. 68, no. 12, pp. 7811–7817, Feb. 2023, doi: 10.1109/tac.2023.3246773.

[52] H. Gyotoku, T. Yuno, Y. Ebihara, R. Tempo, and S. Tarbouriech, "Detecting Destabilizing Nonlinearities in Absolute Stability Analysis with Static O'Shea Zames Falb Multipliers," pp. 4633–4638, Dec. 2025, doi: 10.1109/cdc57313.2025.11312457.

[53] D. Dragičević, A. L. Sasu, B. Sasu, and A. Şirianţu, "Zabczyk Type Criteria for Asymptotic Behavior of Dynamical Systems and Applications," *Journal of Dynamics and Differential Equations* , Aug. 2023, doi: 10.1007/s10884 023 10303 0.

[54] A. Botta, E. Caverzasi, and A. Russo, "When complexity meets finance: A contribution to the study of the macroeconomic effects of complex financial systems," *Research Policy* , vol. 51, no. 8, pp. 103990–103990, May 2020, doi: 10.1016/j.respol.2020.103990.

[55] A. Polyakov, D. Efimov, and X. Ping, "Consistent discretization of homogeneous finite/fixed time controllers for LTI systems," *Automatica* , vol. 155, pp. 111118–111118, Jun. 2023, doi: 10.1016/j.automatica.2023.111118.

[56] M. Tranninger, R. Seeber, M. Steinberger, M. Horn, and C. Pötzsche, "Detectability Conditions and State Estimation for Linear Time Varying and Nonlinear Systems," *SIAM Journal on Control and Optimization* , vol. 60, no. 4, pp. 2514–2537, Aug. 2022, doi: 10.1137/21m1429473.

[57] L. Baratchart, S. Fueyo, and J. Pomet, “Exponential stability of periodic difference delay systems and 1 D hyperbolic PDEs of conservation laws,” *IFAC PapersOnLine* , vol. 55, no. 36, pp. 228–233, Jan. 2022, doi: 10.1016/j.ifacol.2022.11.362.

[58] B. Song *et al.* , “Reliability Analysis of Large Scale Adaptive Weighted Networks,” *IEEE Transactions on Information Forensics and Security* , vol. 15, pp. 651–665, Jul. 2019, doi: 10.1109/tifs.2019.2926193.

[59] J. Gu, A. Zegeling, and W. Huang, “Bifurcation of Limit Cycles and Isochronous Centers on Center Manifolds for a Class of Cubic Kolmogorov Systems in $${\mathbb {R}}^{3}$$,” *Qualitative Theory of Dynamical Systems* , vol. 22, no. 2, Jan. 2023, doi: 10.1007/s12346 023 00745 8.

[60] J. Sánchez and M. Net, “Numerical continuation methods for large scale dissipative dynamical systems,” *The European Physical Journal Special Topics* , vol. 225, pp. 2465–2486, Nov. 2016, doi: 10.1140/epjst/e2015 50317 2.

[61] P. Dandekar, A. Goel, and D. T. Lee, “Biased assimilation, homophily, and the dynamics of polarization,” *Proceedings of the National Academy of Sciences* , vol. 110, no. 15, pp. 5791–5796, Mar. 2013, doi: 10.1073/pnas.1217220110.

[62] J. D. Crawford, “Introduction to bifurcation theory,” *Reviews of Modern Physics* , vol. 63, no. 4, pp. 991–1037, Oct. 1991, doi: 10.1103/revmodphys.63.991.

[63] J. Coron, “Linearized Control Systems and Applications to Smooth Stabilization,” *SIAM Journal on Control and Optimization* , vol. 32, no. 2, pp. 358–386, Mar. 1994, doi: 10.1137/s0363012992226867.

[64] R. Zhou, T. Quartz, H. D. Sterck, and J. Liu, “Neural Lyapunov Control of Unknown Nonlinear Systems with Stability Guarantees,” *arXiv (Cornell University)* , Jun. 2022, doi: 10.48550/arxiv.2206.01913.

[65] B. A. Brandin, R. Malik, and P. Malik, “Incremental Verification and Synthesis of Discrete Event Systems Guided by Counter Examples,” *IEEE Transactions on Control Systems Technology* , vol. 12, no. 3, pp. 387–401, May 2004, doi: 10.1109/tcst.2004.824795.

[66] A. Jena, D. Kalathil, and L. Xie, “Meta Learning Based Adaptive Stability Certificates for Dynamical Systems,” in *Proceedings of the AAAI Conference on Artificial Intelligence* , Association for the Advancement of Artificial Intelligence, Mar. 2024, pp. 12801–12809. doi: 10.1609/aaai.v38i11.29176.

[67] I. Karafyllis and Z. Jiang, “Necessary and sufficient Lyapunov like conditions for robust nonlinear stabilization,” *ESAIM Control Optimisation and Calculus of Variations* , vol. 16, no. 4, pp. 887–928, Aug. 2009, doi: 10.1051/cocv/2009029.

[68] N. A. Şahin and M. Ö. Yatak, “Fuzzy Backstepping Control of Industrial Liquid Level System,” *Gazi Journal of Engineering Sciences* , vol. 10, no. 2, pp. 355–367, Aug. 2024, doi: 10.30855/gmbd.0705a08.

[69] A. S. Davydov and F. Bullo, "Perspectives on Contractivity in Control, Optimization, and Learning," *IEEE Control Systems Letters* , vol. 8, pp. 2087–2098, Jan. 2024, doi: 10.1109/lcsys.2024.3436127.

[70] J. Banaś and K. Sadarangani, "Compactness Conditions in the Study of Functional, Differential, and Integral Equations," *Abstract and Applied Analysis* , vol. 2013, pp. 1–14, Jan. 2013, doi: 10.1155/2013/819315.

[71] C. Esteve, B. Geshkovski, D. Pighin, and E. Zuazua, "Turnpike in Lipschitz nonlinear optimal control," *HAL (Le Centre pour la Communication Scientifique Directe)* , Nov. 2021, Accessed: Aug. 2025. [Online]. Available: https://hal.archives ouvertes.fr/hal 03018529

[72] M. A. Pinsky, "A New Approach to Reducing Vector Delay Nonlinear Systems: Boundedness and Stability Analysis," *arXiv (Cornell University)* , Sep. 2024, doi: 10.48550/arxiv.2410.20674.

[73] R. Dian, "Global Exponential Stability of a Class of Interconnected Large Scale Systems with Time Delays," Jan. 2005, Accessed: Nov. 2025. [Online]. Available: https://en.cnki.com.cn/Article_en/CJFDTOTAL XNJT20050300C.htm

[74] G. Portilla, M. Bajodek, and S. Mondié, "Necessary and sufficient stability conditions for neutral type delay systems: polynomial approximations," *Mathematics of Control Signals and Systems* , Oct. 2025, doi: 10.1007/s00498 025 00426 8.

[75] A. Ibragimov, "Generalized analysis of methods for solving systems of nonlinear equations with point and interval coefficients," *Numerical Algorithms* , Jul. 2024, doi: 10.1007/s11075 024 01891 z.

[76] Z. Zeng, "A Newton's Iteration Converges Quadratically to Nonisolated Solutions Too," *arXiv (Cornell University)* , Feb. 2022, doi: 10.48550/arxiv.2101.09180.

[77] A. E. Bernardini and O. Bertolami, "Distorted stability pattern and chaotic features for quantized prey predator like dynamics," *Physical review. E* , vol. 107, no. 4, Apr. 2023, doi: 10.1103/physreve.107.044201.

[78] V. Cibulka, M. Korda, and T. Haniš, "Spatio Temporal Decomposition of Sum of Squares Programs for the Region of Attraction and Reachability," *IEEE Control Systems Letters* , vol. 6, pp. 812–817, Jun. 2021, doi: 10.1109/lcsys.2021.3086585.

[79] M. Posa, T. Koolen, and R. Tedrake, "Balancing and Step Recovery Capturability via Sums of Squares Optimization," Jul. 2017, doi: 10.15607/rss.2017.xiii.032.

[80] P. Giesl and S. Hafstein, "Triangulations and Error Estimates for Interpolating Lyapunov Functions," *SN Computer Science* , vol. 4, no. 4, Apr. 2023, doi: 10.1007/s42979 023 01777 2.

[81] A. Priyambodo, H. Rakhmad, and M. Shakir, “Computational Modeling and Simulation of Nonlinear Dynamical System Stability in Applied Mathematics,” *International Journal of Applied Mathematics and Computing.* , vol. 2, no. 2, pp. 16–22, Apr. 2025, doi: 10.62951/ijamc.v2i2.271.

[82] F. Fröhlich, C. Loos, and J. Hasenauer, “Scalable Inference of Ordinary Differential Equation Models of\n Biochemical Processes,” *arXiv (Cornell University)* , Nov. 2017, doi: 10.48550/arxiv.1711.08079.

[83] H. Resat, L. Petzold, and M. F. Pettigrew, “Kinetic Modeling of Biological Systems,” *Methods in molecular biology* . Springer Science+Business Media, pp. 311–335, Jan. 01, 2009. doi: 10.1007/978 1 59745 243 4_14.

[84] G. Datseris and A. Wagemakers, “Effortless estimation of basins of attraction,” *arXiv (Cornell University)* , Oct. 2021, Accessed: Mar. 2025. [Online]. Available: http://arxiv.org/abs/2110.04358

[85] R. C. Calleja *et al.* , “Boundary value problem formulations for computing invariant manifolds and connecting orbits in the circular restricted three body problem,” *Celestial Mechanics and Dynamical Astronomy* , vol. 114, pp. 77–106, Aug. 2012, doi: 10.1007/s10569 012 9434 y.

[86] P. Brzeski and P. Perlikowski, “Sample Based Methods of Analysis for Multistable Dynamical Systems,” *Archives of Computational Methods in Engineering* , vol. 26, no. 5, pp. 1515–1545, Sep. 2018, doi: 10.1007/s11831 018 9280 5.

[87] C. Kao, A. Megretski, U. Jönsson, and A. Rantzer, “A MATLAB toolbox for robustness analysis,” vol. 37, pp. 297–302, Mar. 2005, doi: 10.1109/cacsd.2004.1393892.

[88] A. Papachristodoulou *et al.* , “SOSTOOLS Version 4.00 Sum of Squares Optimization Toolbox for MATLAB,” *arXiv (Cornell University)* , Oct. 2013, doi: 10.48550/arxiv.1310.4716.

[89] M. Lubin and I. Dunning, “Computing in Operations Research Using Julia,” *INFORMS journal on computing* , vol. 27, no. 2, pp. 238–248, Mar. 2015, doi: 10.1287/ijoc.2014.0623.

[90] D. M. Lopez *et al.* , “ARCH COMP22 Category Report: Artificial Intelligence and Neural Network Control Systems (AINNCS) for Continuous and Hybrid Systems Plants,” *EPiC series in computing* , vol. 90, pp. 142–98, Dec. 2022, doi: 10.29007/wfgr.

[91] D. Henrion and M. Korda, “Convex Computation of the Region of Attraction of Polynomial Control Systems,” *IEEE Transactions on Automatic Control* , vol. 59, no. 2, pp. 297–312, Sep. 2013, doi: 10.1109/tac.2013.2283095.

[92] A. K. Alekseev, A. E. Бондарев, В. А. Галактионов, A. E. Kuvshinnikov, and L. Z. Shapiro, “On Applying of Generalized Computational Experiment to Numerical Methods Verification,” *Proceedings of the 30th International Conference on Computer Graphics and Machine Vision (GraphiCon 2020) Part 1* , Dec. 2020, doi: 10.51130/graphicon 2020 2 3 19.

[93] D. Shevitz and B. Paden, “Lyapunov stability theory of nonsmooth systems,” *IEEE Transactions on Automatic Control* , vol. 39, no. 9, pp. 1910–1914, Jan. 1994, doi: 10.1109/9.317122.

[94] U. Mandal *et al.* , “Formally Verifying Deep Reinforcement Learning Controllers with Lyapunov Barrier Certificates,” *arXiv (Cornell University)* , May 2024, doi: 10.48550/arxiv.2405.14058.

[95] M. H. Meng *et al.* , “Adversarial Robustness of Deep Neural Networks: A Survey from a Formal Verification Perspective,” *IEEE Transactions on Dependable and Secure Computing* , pp. 1–1, May 2022, doi: 10.1109/tdsc.2022.3179131.

[96] J. Köhler, M. N. Zeilinger, and L. Grüne, “Stability and Performance Analysis of NMPC: Detectable Stage Costs and General Terminal Costs,” *IEEE Transactions on Automatic Control* , vol. 68, no. 10, pp. 6114–6129, Jan. 2023, doi: 10.1109/tac.2023.3235244.

[97] M. Johansson and H. Taghavian, “Stable MPC with maximal terminal sets and quadratic terminal costs,” *arXiv (Cornell University)* , Jun. 2024, doi: 10.48550/arxiv.2406.02760.

[98] Z. Xiong, H. Wang, L. Zhao, and A. Papachristodoulou, “Data Driven Stable Neural Feedback Loop Design,” *arXiv (Cornell University)* , May 2024, doi: 10.48550/arxiv.2405.02100.

[99] C. L. Galimberti, L. Furieri, and G. Ferrari-Trecate, “Parametrizations of All Stable Closed loop Responses: From Theory to Neural Network Control Design,” *arXiv (Cornell University)* , Dec. 2024, doi: 10.48550/arxiv.2412.19280.

[100] N. Cangiotti, M. Capolli, M. Sensi, and S. Sottile, “A survey on Lyapunov functions for epidemic compartmental models,” *Bollettino dell Unione Matematica Italiana* , vol. 17, no. 2, pp. 241–257, Jun. 2023, doi: 10.1007/s40574 023 00368 6.

[101] W. Wu, J. Chen, and J. Chen, “Stability Analysis of Systems with Recurrent Neural Network Controllers,” *IFAC PapersOnLine* , vol. 55, no. 12, pp. 170–175, Jan. 2022, doi: 10.1016/j.ifacol.2022.07.306.

[102] M. Roşca, Y. Wu, C. Qin, and B. Dherin, “On a continuous time model of gradient descent dynamics and instability in deep learning,” *arXiv (Cornell University)* , Feb. 2023, doi: 10.48550/arxiv.2302.01952.

[103] N. Buxton, G. Crăciun, A. Deshpande, and C. Pantea, “Weakly reversible deficiency zero realizations of reaction networks,” *arXiv (Cornell University)* , Feb. 2025, doi: 10.48550/arxiv.2502.17461.

[104] H. Damak, “Input to state stability and integral input to state stability of non autonomous infinite dimensional systems,” *International Journal of Systems Science* , vol. 52, no. 10, pp. 2100–2113, Feb. 2021, doi: 10.1080/00207721.2021.1879306.

[105] J. Wang, C. Schlosser, M. Korda, and V. Magron, “Exploiting Term Sparsity in Moment SOS Hierarchy for Dynamical Systems,” *IEEE Transactions on Automatic Control* , vol. 68, no. 12, pp. 8232–8237, Jul. 2023, doi: 10.1109/tac.2023.3293014.

[106] A. S. Davydov, S. Jafarpour, and F. Bullo, “Non Euclidean Contraction Theory via Semi Inner Products.,” *arXiv (Cornell University)* , Mar. 2021, Accessed: Oct. 2025. [Online]. Available: http://export.arxiv.org/pdf/2103.12263

[107] D. Grande, A. Peruffo, E. Anderlini, and G. Salavasidis, “Augmented Neural Lyapunov Control,” *IEEE Access* , vol. 11, pp. 67979–67986, Jan. 2023, doi: 10.1109/access.2023.3291349.

[108] H. Zhao, B. Liu, L. Dehbi, H. Xie, Z. Yang, and H. Qian, “Polynomial Neural Barrier Certificate Synthesis of Hybrid Systems via Counterexample Guidance,” *IEEE Transactions on Computer Aided Design of Integrated Circuits and Systems* , vol. 43, no. 11, pp. 3756–3767, Nov. 2024, doi: 10.1109/tcad.2024.3447226.

[109] J. O. Achuobi, E. P. Akpan, R. George, and A. E. Ofem, “Stability analysis of Caputo fractional time dependent systems with delay using vector lyapunov functions,” *AIMS Mathematics* , vol. 9, no. 10, pp. 28079–28099, Jan. 2024, doi: 10.3934/math.20241362.

[110] F. T. Sánchez, A. Raimundo, S. Cerda, P. Pablo, and C. Alzate, “Stability Analysis of a Lotka Volterra Type Biological Model in Its Fractional Version,” *Journal of Hunan University Natural Sciences* , vol. 50, no. 10, Jan. 2023, doi: 10.55463/issn.1674 2974.50.10.9.

[111] Y. Tian, Z. Wang, and C. Wang, “Stability and stabilisation for time varying delay systems based on flexible augmented Lyapunov–Krasovskii functional,” *International Journal of Control* , vol. 97, no. 4, pp. 714–727, Jan. 2023, doi: 10.1080/00207179.2023.2168112.

[112] D. Efimov and A. Yu. Aleksandrov, “On Equivalence of Lyapunov–Razumikhin Conditions and ISS for a Class of Time Delay Systems,” *IEEE Transactions on Automatic Control* , vol. 69, no. 8, pp. 5666–5671, Mar. 2024, doi: 10.1109/tac.2024.3382609.

[113] J. Liu, Y. Meng, M. Fitzsimmons, and R. Zhou, “Compositionally Verifiable Vector Neural Lyapunov Functions for Stability Analysis of Interconnected Nonlinear Systems,” *arXiv (Cornell University)* , Mar. 2024, doi: 10.48550/arxiv.2403.10007.

[114] S. A. Deka, A. M. Valle, and C. J. Tomlin, “Koopman based Neural Lyapunov functions for general attractors,” in *2022 IEEE 61st Conference on Decision and Control (CDC)* , Dec. 2022, pp. 5123–5128. doi: 10.1109/cdc51059.2022.9992927.

[115] A. R. Tavakolpour Saleh, “A novel theorem on motion stability,” *Chaos Solitons & Fractals* , vol. 153, pp. 111526–111526, Nov. 2021, doi: 10.1016/j.chaos.2021.111526.

[116] A. Felski and F. K. Kunst, “Exceptional Points and Stability in Nonlinear Models of Population Dynamics having $\mathcal{PT}$ symmetry,” *arXiv (Cornell University)* , Nov. 2024, doi: 10.48550/arxiv.2411.12167.

[117] R. M. Bora, B. S. Dey, and I. N. Kar, "On the Contraction Analysis of Nonlinear System with Multiple Equilibrium Points," *arXiv (Cornell University)* , Feb. 2025, doi: 10.48550/arxiv.2502.14242.

[118] C. Wu and D. V. Dimarogonas, "From Partial and Horizontal Contraction to k Contraction," *IEEE Transactions on Automatic Control* , vol. 69, no. 4, pp. 2391–2398, Aug. 2023, doi: 10.1109/tac.2023.3303037.

[119] S. Jafarpour, A. S. Davydov, and F. Bullo, "Non Euclidean Contraction Theory for Monotone and Positive Systems," *IEEE Transactions on Automatic Control* , vol. 68, no. 9, pp. 5653–5660, Nov. 2022, doi: 10.1109/tac.2022.3224094.

[120] A. A. Maruf, S. Kundu, E. Yeung, and M. Anghel, "Decomposition of Nonlinear Dynamical Networks via Comparison Systems," *arXiv (Cornell University)* , Mar. 2022, doi: 10.48550/arxiv.1803.02967.

[121] T. X. Nghiem *et al.* , "Physics Informed Machine Learning for Modeling and Control of Dynamical Systems," in *2022 American Control Conference (ACC)* , May 2023. doi: 10.23919/acc55779.2023.10155901.

[122] Y. Schiff *et al.* , "DySLIM: Dynamics Stable Learning by Invariant Measure for Chaotic Systems," *arXiv (Cornell University)* , Feb. 2024, doi: 10.48550/arxiv.2402.04467.

[123] J. Cheng, R. Wang, and I. R. Manchester, "Learning Stable and Passive Neural Differential Equations," *arXiv (Cornell University)* , Apr. 2024, doi: 10.48550/arxiv.2404.12554.

[124] J. Chaudhary, D. K. Nidhi, J. Heikkonen, H. Merisaari, and R. Kanth, "Super Level Sets and Exponential Decay: A Synergistic Approach to Stable Neural Network Training," *arXiv (Cornell University)* , Sep. 2024, doi: 10.48550/arxiv.2409.16769.

[125] H. Zhang, D. Yu, M. Yi, W. Chen, and T. Liu, "Stabilize Deep ResNet with A Sharp Scaling Factor $τ$," *arXiv (Cornell University)* , Feb. 2022, doi: 10.48550/arxiv.1903.07120.

[126] D. S. Kushwaha and Z. A. Biron, "A Review On Safe Reinforcement Learning Using Lyapunov and Barrier Functions," *arXiv (Cornell University)* . Cornell University, Aug. 12, 2025. doi: 10.48550/arxiv.2508.09128.

[127] W. Ren and J. Xiong, "Stability and $\mathcal{H}_{\infty}$ Performance Analysis of Stochastic Linear Networked and Quantized Control Systems," *arXiv (Cornell University)* , Feb. 2022, doi: 10.48550/arxiv.2103.02894.

[128] M. Hertneck and F. Allgöwer, "Robust dynamic self triggered control for nonlinear systems using hybrid Lyapunov functions," *arXiv (Cornell University)* , Nov. 2021, doi: 10.48550/arxiv.2111.04347.

[129] A. Mironchenko, "Input to state stability meets small gain theory," *arXiv (Cornell University)* , Jun. 2024, doi: 10.48550/arxiv.2406.17909.

[130] G. Bet, F. Coppini, and F. R. Nardi, “Weakly interacting oscillators on dense random graphs,” *Journal of Applied Probability* , vol. 61, no. 1, pp. 255–278, Jun. 2023, doi: 10.1017/jpr.2023.34.

[131] F. Höfer and H. M. Soner, “Synchronization Games,” *arXiv (Cornell University)* , Feb. 2024, doi: 10.48550/arxiv.2402.08842.

[132] M. Nourian and P. E. Caines, “ε Nash Mean Field Game Theory for Nonlinear Stochastic Dynamical Systems with Major and Minor Agents,” *arXiv (Cornell University)* , Mar. 2022, doi: 10.48550/arxiv.1209.5684.

[133] M. Beisenbi and S. Kaliyeva, “Approach to the synthesis of an aperiodic robust automatic control system based on the gradient speed method of Lyapunov vector functions,” *Eastern European Journal of Enterprise Technologies* , vol. 1, pp. 6–14, Feb. 2023, doi: 10.15587/1729 4061.2023.274063.

[134] A. M. Zenkin, A. Peregudin, and A. Bobtsov, “Lyapunov function search method for analysis of nonlinear systems stability using genetic algorithm,” *Scientific and technical journal of information technologies mechanics and optics* , vol. 23, no. 5, pp. 886–893, Oct. 2023, doi: 10.17586/2226 1494 2023 23 5 886 893.

[135] S. Chen, M. Fazlyab, M. Morari, G. J. Pappas, and V. M. Preciado, “Learning Lyapunov Functions for Piecewise Affine Systems with Neural Network Controllers,” *arXiv (Cornell University)* , Feb. 2022, doi: 10.48550/arxiv.2008.06546.

[136] W. Xv, H. Wan, X. Luan, and F. Liu, “Global Stability Ensured Controller Design for Nonlinear Markov Jump Systems Based on Contraction Theory,” *Research Square (Research Square)* , Nov. 2024, doi: 10.21203/rs.3.rs 5408578/v1.

[137] Y. Lai, “Finding nonlinear system equations and complex network structures from data: A sparse optimization approach,” *Chaos An Interdisciplinary Journal of Nonlinear Science* , vol. 31, no. 8, Aug. 2021, doi: 10.1063/5.0062042.

[138] A. Pal, S. Bhowmick, and S. Nagarajaiah, “Physics informed AI and ML based sparse system identification algorithm for discovery of PDE’s representing nonlinear dynamic systems,” *arXiv (Cornell University)* , Oct. 2024, doi: 10.48550/arxiv.2410.10023.

[139] C. Schlosser and M. Korda, “Sparse decompositions of nonlinear dynamical systems and applications to moment sum of squares relaxations,” *arXiv (Cornell University)* , Feb. 2022, doi: 10.48550/arxiv.2012.05572.

[140] M. Arcak, “Compositional Design and Verification of Large Scale Systems Using Dissipativity Theory: Determining Stability and Performance from Subsystem Properties and Interconnection Structures,” *IEEE Control Systems* , vol. 42, no. 2, pp. 51–62, Mar. 2022, doi: 10.1109/mcs.2021.3139721.

[141] D. Xu and V. Ugrinovskii, “Decentralized Measurement Feedback Stabilization of Large scale Systems via Control Vector Lyapunov Functions,” *arXiv (Cornell University)* , Mar. 2022, doi: 10.48550/arxiv.1310.0607.

[142] A. Mironchenko, “Non uniform ISS small gain theorem for infinite networks,” *IMA Journal of Mathematical Control and Information* , vol. 38, no. 4, pp. 1029–1045, Aug. 2021, doi: 10.1093/imamci/dnab026.

[143] V. Erofeeva and S. Parsegov, “Properties of the Laplacian spectra of certain basic and hierarchical graphs,” *Cybernetics and Physics* , pp. 12–19, Jun. 2024, doi: 10.35470/2226 4116 2024 13 1 12 19.

[144] P. Florchinger, “Stabilization of partially linear composite stochastic systems via stochastic Luenberger observers,” *Kybernetika* , pp. 626–636, Nov. 2022, doi: 10.14736/kyb 2022 4 0626.

[145] M. M. Shumafov and V. B. Tlyachev, “Stochastic stability criteria for two dimensional linear autonomous systems perturbed by white noise,” *arXiv (Cornell University)* , Apr. 2021, Accessed: Oct. 2025. [Online]. Available: http://arxiv.org/abs/2104.01637

[146] P. den Boef, D. Manvelyan, J. M. L. Maubach, W. H. A. Schilders, and N. van de Wouw, “Stable Sparse Operator Inference for Nonlinear Structural Dynamics,” *arXiv (Cornell University)* , Jul. 2024, doi: 10.48550/arxiv.2407.21672.

[147] E. Ibargüen Mondragón, M. V. Ortega, and S. Hidalgo-Bonilla, “On the orbital stability of traveling waves that behave such as particles,” *Revista Boletín Redipe* , vol. 10, no. 2, pp. 158–164, Feb. 2021, doi: 10.36260/rbr.v10i2.1202.

[148] H. Tsukamoto, S. Chung, J. J. Slotine, and C. Fan, “A Theoretical Overview of Neural Contraction Metrics for Learning based Control with Guaranteed Stability,” *arXiv (Cornell University)* , Oct. 2021, doi: 10.48550/arxiv.2110.00693.

[149] H. Zou, J. Feng, H. Zhao, and Y. Shi, “Analytical Lyapunov Function Discovery: An RL based Generative Approach,” *arXiv (Cornell University)* , Feb. 2025, doi: 10.48550/arxiv.2502.02014.

[150] A. Gahlawat, S. H. Karumanchi, and N. Hovakimyan, “$\mathcal{L}_1$ DRAC: Distributionally Robust Adaptive Control,” *arXiv (Cornell University)* , Sep. 2025, doi: 10.48550/arxiv.2509.04619.

[151] R. Sinha, J. M. Harrison, S. M. Richards, and M. Pavone, “Adaptive Robust Model Predictive Control with Matched and Unmatched Uncertainty,” in *2022 American Control Conference (ACC)* , Jun. 2022. doi: 10.23919/acc53348.2022.9867457.

[152] V. Zinage, S. P. A. Ram, M. R. Akella, and E. Bakolas, “Semiglobal Exponential Stability for Dual Quaternion Based Rigid Body Tracking Control,” *Journal of Guidance Control and Dynamics* , pp. 1–14, Jun. 2024, doi: 10.2514/1.g008323.

[153] I. Gracia, D. Boskos, M. Lahijanian, L. Laurenti, and M. Mazo, “Efficient strategy synthesis for switched stochastic systems with distributional uncertainty,” *Nonlinear Analysis Hybrid Systems* , vol. 55, pp. 101554–101554, Nov. 2024, doi: 10.1016/j.nahs.2024.101554.

[154] L. Cui, Z. Jiang, and E. D. Sontag, “Small Covariance Noise to State Stability of Stochastic Systems and Its Applications to Stochastic Gradient Dynamics,” *arXiv (Cornell University)* , Sep. 2025, doi: 10.48550/arxiv.2509.24277.