

Continuous Mappings and Their Generalisations in Metric and Topological Spaces: A Review of Semi-, Semi-Generalised, Uniform and Separate Continuity

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Abstract. The notion of a continuous mapping is the organising idea of general topology: it turns a family of open sets into a category and links the internal structure of a space to the behaviour of maps defined on it. This article surveys the literature on continuous mappings between metric and topological spaces, from the classical formulations of Fréchet, Hausdorff and Kuratowski to the large body of generalised continuities that grew from the work of Levine and Njåstad. Particular attention is given to semi-continuity and semi-generalised continuity, to the problem of separate versus joint continuity, in which quasi-continuity is the central tool, and to uniform continuity and its metric refinements. For each class of maps we trace the properties that are preserved or reflected, namely compactness, connectedness, convergence, separation and metrizability. A critical analysis identifies recurring structural difficulties: proliferation of named classes, failure of closure under composition, repetition of results derivable from minimal structures, weak contact with metric phenomena, and neglect of function spaces. Eight open problems are formulated.

Keywords: continuity; semi-continuity; generalised closed sets; sg-continuity; quasi-continuity; separate continuity; uniform continuity; minimal structures; separation axioms; metrization

1. Introduction

The notion of a continuous mapping is the single most important organising idea in topology. A topology on a set specifies which subsets are to be regarded as “open”; it is the class of continuous maps that converts such specifications into a *category*, and hence into a usable mathematical theory. Almost every substantive theorem in general topology can be read as a statement about the interaction between (i) the internal structure of a space (separation, compactness, connectedness, countability, metrizability) and (ii) the behaviour of maps into or out of that space.

The classical treatises of Kelley [1], Willard [2], Engelking [3] and Munkres [4] remain the standard references for the core theory, and the catalogue of Steen and Seebach [5] supplies the counterexamples that delimit it. A more recent reorganisation of the subject around convergence rather than open sets is given by Dolecki and Mynard [6]; this viewpoint is relevant below because several generalised continuities are most transparently described through filters and nets rather than inverse images of open sets.

The article is organised as follows. Section 2 traces the emergence of the classical definition of continuity and its standard equivalent formulations. Section 3 reviews the literature on generalised continuous mappings, including the “generalised closed set” programme. Sections 4 and 5 treat semi-continuity and semi-generalised continuity, Section 6 separate versus joint continuity, Section 7 uniform continuity, Section 8 compactness, connectedness, convergence and function spaces, and Section 9 separation axioms and metrization. Section 10 gives a critical appraisal and formulates open problems, and Section 11 concludes.

Throughout, (X, τ) , (Y, σ) , (Z, η) denote topological spaces on which no separation axiom is assumed unless stated; $\text{cl}(A)$ and $\text{int}(A)$ denote closure and interior; (X, d) is a metric space; and all maps are functions in the set-theoretic sense. Where a notion was introduced independently under different names, the earliest traceable attribution is given.

2. Classical continuous mappings

2.1 Origins and equivalent formulations

The ε - δ definition of continuity for real functions, due in substance to Bolzano and Cauchy and sharpened by Weierstrass, was the point of departure. Its decisive abstraction occurred when Fréchet [7] defined continuity of a mapping between sets

with an abstract notion of sequential limit as preservation of limits. The neighbourhood-based abstraction appears in Hausdorff [8], where a map is continuous at a point when every neighbourhood of the image contains the image of some neighbourhood of the point. Kuratowski [9] replaced neighbourhoods by a closure operator and obtained the formulation that $f: X \rightarrow Y$ is continuous if and only if

$$f(\text{cl}_X(A)) \subseteq \text{cl}_Y(f(A)) \text{ for all } A \subseteq X.$$

This closure characterisation is of more than aesthetic interest: almost every weakened continuity discussed below is obtained by replacing one or both closure operators in this inclusion by a weaker operator (semi-closure, pre-closure, α -closure, δ -closure). The modern inventory of equivalent conditions — $f^{-1}(V)$ open for all open V ; $f^{-1}(F)$ closed for all closed F ; $\text{cl } f^{-1}(B) \subseteq f^{-1}(\text{cl } B)$; continuity at each point in the neighbourhood sense; preservation of convergence of nets or filters — is assembled in Kelley [1], Bourbaki [10], Dugundji [11] and Sierpiński [12]. The equivalence of the global and pointwise formulations permitted continuity to be studied without reference to points, and hence permitted the lattice-theoretic and categorical treatments.

2.2 Structural theorems

Four groups of classical results delimit what continuity can and cannot do, and they recur as benchmarks throughout the later literature. *Preservation theorems*: continuous images of compact, connected, separable and Lindelöf spaces are again compact, connected, separable and Lindelöf; the compactness result is explicit in Alexandroff and Urysohn [13]. Continuity does not, however, preserve metrizability, local compactness, first countability or any separation axiom above T_0 without further hypotheses, a fact systematically exploited by Steen and Seebach [5]. *Reflection theorems*: if Y is Hausdorff the graph of a continuous f is closed; if moreover X is compact, f is a closed map and a continuous bijection is a homeomorphism. This last result is the prototype of all “automatic improvement” theorems: one asks whether a weak continuity plus a compactness-type hypothesis forces full continuity.

Extension and factorisation theorems: the Urysohn lemma [14] and Tietze extension theorem [15] show that normality is exactly the hypothesis permitting continuous real-valued functions to separate closed sets and to be extended from closed subspaces; Tychonoff [16] showed that complete regularity is exactly the hypothesis permitting embedding into a cube, leading to the Čech–Stone compactification [17]. *Initial and final structures*: the characterisations of continuity for maps into products and out of quotients [10] identify the subspace, product, quotient and sum topologies, and the Tychonoff product theorem [16], with Alexander’s subbase lemma [18], belongs here. Generalised open sets, by contrast, are in general badly behaved under products, so the generalised continuous maps usually fail to form a category with initial and final structures — a deficiency taken up in Section 10.

2.3 Convergence formulations

Moore and Smith [19] introduced nets and showed that continuity is equivalent to preservation of net convergence in arbitrary spaces; Birkhoff [20] developed the consequences, and Cartan [21] introduced filters and ultrafilters, yielding the parallel theory used by Bourbaki [10]. Sequential criteria for continuity are valid precisely in sequential spaces, while in metric spaces continuity, sequential continuity and the Fréchet limit-preservation condition coincide; any comparison of metric and general topological phenomena must distinguish these carefully. The convergence formulation also extends to weak continuities: θ -continuity, δ -continuity and almost continuity have clean characterisations in terms of filters and the θ - and δ -closure operators of Veličko [22].

2.4 Strong forms of continuity

Alongside weakenings stands an equally long tradition of strengthenings. Levine [23] introduced *strong continuity* ($f(\text{cl } A) \subseteq f(A)$ for all A , equivalently $f^{-1}(B)$ clopen for all $B \subseteq Y$); Arya and Gupta [24] examined strongly and *completely continuous* maps (inverse images of open sets are regular open in the sense of Stone [25]); and Noiri [26] introduced *super-continuity* ($f^{-1}(V)$ is δ -open for open V). The resulting hierarchy is

$$\text{strongly continuous} \Rightarrow \text{perfectly continuous} \Rightarrow \text{completely continuous} \Rightarrow \text{super-continuous} \Rightarrow \text{continuous}.$$

The strong side of the hierarchy is far better organised than the weak side, because each strong class is defined by replacing “open” with a canonically defined smaller family (regular open, δ -open, clopen), whereas the weak side involves many mutually incomparable larger families.

3. Generalised continuous mappings

3.1 The two generating ideas

Virtually all generalised continuities arise from one of two moves. **Move I** weakens the open sets: fix an operator $\square$ assigning to each space a family of sets containing τ , and call f $\square$ -continuous if $f^{-1}(V) \in \square(X)$ for every $V \in \sigma$. Taking $\square$ to be the semi-open sets of Levine [27], the pre-open sets of Mashhour, Abd El-Monsef and El-Deeb [28], the α -sets of Njåstad [29], the β -open (semi-pre-open) sets [30], [31] or the b -open sets of Andrijević [32] yields semi-, pre-, α - [33], β - and b -continuity. **Move II** weakens the inverse-image condition itself, asking only for an inclusion between closures or interiors. Levine's *weak continuity* [34] requires $f^{-1}(V) \subseteq \text{int } f^{-1}(\text{cl } V)$ for open V ; Husain [35] and Singal and Singal [36] introduced two distinct notions of *almost continuity*; Veličko's closures [22] generate θ - and δ -continuity [37]; Rose [38] clarified the relations among them; and Gentry and Hoyle [39] introduced the extremely weak *somewhat continuity*, which nevertheless suffices for certain Baire-type conclusions. Table 1 summarises the principal classes.

Table 1. Principal generalised continuities and their defining conditions.

Class	Defining condition on $f:X \rightarrow Y$	Principal sources
Semi-continuity	$f^{-1}(V)$ semi-open for open V	[27]
Pre-continuity	$f^{-1}(V)$ pre-open for open V	[28]
α -continuity	$f^{-1}(V)$ an α -set for open V	[29], [33]
β - and b -continuity	$f^{-1}(V)$ semi-pre-open, respectively b -open	[30], [31], [32]
Weak continuity	$f^{-1}(V) \subseteq \text{int } f^{-1}(\text{cl } V)$ for open V	[34]
Almost continuity	two distinct notions	[35], [36]
θ -, δ -continuity	continuity relative to θ - and δ -closures	[22], [37]
Somewhat continuity	$f^{-1}(V) \neq \emptyset$ implies $\text{int } f^{-1}(V) \neq \emptyset$	[39]
g -continuity	$f^{-1}(F)$ g -closed for closed F	[40]
sg- and gs-continuity	$f^{-1}(F)$ sg-closed, respectively gs-closed	[41], [42]

Njåstad [29] proved that the α -sets form a topology τ_α finer than τ with the same semi-open sets, and that

$$\tau \subseteq \tau_\alpha = SO(X) \cap PO(X) \subseteq SO(X) \cup PO(X) \subseteq \beta O(X),$$

with b -open sets lying between $SO(X) \cup PO(X)$ and $\beta O(X)$ [32]. Correspondingly, continuity $\Rightarrow \alpha$ -continuity $\Rightarrow$ semi-continuity and pre-continuity $\Rightarrow b$ -continuity $\Rightarrow \beta$ -continuity, all implications strict and the two middle classes incomparable. Decomposition theorems, for example α -continuity = semi-continuity + pre-continuity, are the characteristic results of this literature and go back in spirit to Levine's decomposition of continuity [34].

3.2 Generalised closed sets

A historically distinct programme begins with Levine [43], who called $A \subseteq X$ *generalised closed* (g -closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open. The motivation was a new separation axiom strictly between T_0 and T_1 : X is $T_{1/2}$ if and only if every g -closed set is closed, and Dunham [44] proved that this holds if and only if every singleton is open or closed. Balachandran, Sundaram and Maki [40] introduced g -continuous maps and began the systematic study of g -open and g -closed maps and g -homeomorphisms. A proliferation of variants followed, combining a weak open family with the g -closedness scheme: gs-closed sets [42], αg - and $g\alpha$ -closed sets [45], [46], rg-closed sets [47], gsp-closed sets [48], δg -closed sets with the axiom $T_{3/4}$ [49], and gpr-closed sets [50].

The resulting implication diagrams, reproduced with variations in dozens of papers, have a common content: each arrow is strict; the diagram is not a lattice; and the classes are generally not closed under finite unions or intersections, so they do not form topologies, generalised or otherwise. This is the technical root of the difficulties catalogued in Section 10.

3.3 Unification: minimal structures and generalised topologies

Two attempts to impose order on this proliferation are important. Maki [51] introduced *minimal structures*: families $m_X \subseteq \square(X)$ containing $\emptyset$ and X with no closure requirements whatever. Popa and Noiri [52] defined M -continuity between such

spaces and showed that semi-, pre-, α -, β -, b -continuity, almost continuity and weak continuity are all instances, so that their characterisation, preservation and decomposition theorems become corollaries of a single theorem. Császár [53], [54] independently developed *generalised topologies* (families closed under arbitrary unions but not necessarily finite intersections), obtaining a theory in which the interior and closure operators retain monotonicity and idempotence and in which products and function spaces can be discussed.

4. Semi-continuity

Levine [27] called $A \subseteq X$ *semi-open* if there is an open U with $U \subseteq A \subseteq \text{cl}(U)$, equivalently $A \subseteq \text{cl}(\text{int } A)$, and called f *semi-continuous* if $f^{-1}(V)$ is semi-open for every open V . The family $SO(X)$ contains τ and is closed under arbitrary unions but in general not under intersections; the complements are the semi-closed sets. Biswas [55] gave the first systematic characterisations, including the neighbourhood form at a point. Crossley and Hildebrand [56] introduced the semi-closure $\text{scl}(A)$, the smallest semi-closed set containing A , and the semi-interior, showing that scl is monotone, idempotent and expansive but not additive, so that $(X, SO(X))$ is supratopological but not topological; in [57] they introduced semi-homeomorphisms and semi-topological properties.

The *composition problem* is the defining structural pathology of the field. Noiri [58] collected the most-cited composition results: the composition of two semi-continuous maps need not be semi-continuous; if f is semi-continuous and g continuous then $g \circ f$ is semi-continuous while $f \circ g$ need not be; and irresolute maps (inverse images of semi-open sets are semi-open) are exactly the maps composable with semi-continuous maps on the left. Semi-continuous maps therefore do not form the morphisms of a category, and much of the technical literature consists of auxiliary classes (irresolute, pre-semi-open, α -irresolute, M -semicontinuous) that restore composability.

The preservation literature is summarised by four themes. *Connectedness* is preserved by semi-continuous surjections [55], [58], one of the few preservation theorems that survives the passage from continuity to semi-continuity intact. *Compactness* is not; each weak continuity requires its own compactness notion, such as S -closedness (Thompson [59]; Cameron [60]), so that the classical pairing (continuity, compactness) splits into a two-parameter family. For *separation*, Maheshwari and Prasad [61], [62] introduced semi- T_i axioms, and Jankovic and Reilly [63] proved that X is semi- T_1 if and only if every singleton is open or nowhere dense, an exact analogue of Dunham's theorem. Finally, *quasi-continuity*, which arose independently in real analysis, is equivalent to semi-continuity; Neubrunn's survey [64] documents this coincidence and the resulting transfer of results, a point of central importance for Section 6.

5. Semi-generalised continuity

Bhattacharyya and Lahiri [65] defined A to be *semi-generalised closed* (sg-closed) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. Arya and Nour [42], independently, defined *generalised semi-closed* (gs-closed) sets by the same condition with U merely open. The notions differ only in the class of ambient sets, and since $\tau \subseteq SO(X)$ one has

$$\text{closed} \Rightarrow \text{semi-closed} \Rightarrow \text{sg-closed} \Rightarrow \text{gs-closed}, \quad \text{closed} \Rightarrow g\text{-closed} \Rightarrow \text{gs-closed},$$

with sg-closed and g -closed incomparable. Among the basic properties [65], [42]: the union of two sg-closed sets need not be sg-closed; A is sg-closed if and only if $\text{scl}(A) \setminus A$ contains no nonempty semi-closed set; and in extremally disconnected spaces several classes collapse. Bhattacharyya and Lahiri also introduced the axiom semi- $T_{1/2}$ (every sg-closed set is semi-closed) and proved the analogue of the Levine–Dunham theorems.

Sundaram, Maki and Balachandran [41] introduced *sg-continuous* maps and established the fundamental theory: characterisations through sg-open sets and sg-neighbourhoods; the strict implications continuous $\Rightarrow$ semi-continuous $\Rightarrow$ sg-continuous $\Rightarrow$ gs-continuous; failure of closure under composition; behaviour under restriction and graphs; and the characterisation of semi- $T_{1/2}$ spaces as exactly those in which sg-continuity coincides with semi-continuity. Devi, Balachandran and Maki [66], [67] developed sg-closed maps and sg-homeomorphisms, proving that the latter form a group under composition only under additional hypotheses — again isolating composability as the central issue. Caldas [68] obtained decomposition theorems of the form “sg-continuity + an auxiliary condition = semi-continuity”. Maki, Umehara and Noiri [69] showed that every topological space is pre- $T_{1/2}$, which, by contrast with the semi-case, shows that $T_{1/2}$ -type axioms are highly sensitive to the weak open family used and that no uniform theory covers all cases.

Downstream of sg- and gs-closedness lie gsp-, ag -, ga -, rg-, gpr- and δg -closed sets [45], [46], [47], [48], [49], [50], each with a continuity notion, a separation axiom of $T_{1/2}$ type and a family of preservation theorems. The dominant form of a paper in this area is: define the class; place it in the implication diagram with counterexamples; characterise the corresponding continuity; prove the two or three preservation theorems that survive; and identify the separation axiom under which the class collapses to an older one.

6. Separate and joint continuity

A function $f: X \times Y \rightarrow Z$ is *separately continuous* if every partial map $f(x, \cdot)$ and $f(\cdot, y)$ is continuous and *jointly continuous* if it is continuous on the product. Joint continuity implies separate continuity; the converse fails, as the classical example $f(x, y) = xy/(x^2 + y^2)$, $f(0, 0) = 0$, shows. The modern problem — to determine where a separately continuous function must be jointly continuous — begins with Baire [70], who proved that a separately continuous $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is of the first Baire class and hence has a residual set of points of joint continuity, and with Lebesgue [71]; Sierpiński [72] obtained sharper structural results, and Rudin [73] gave a short modern proof of Lebesgue's first theorem.

The decisive general result is due to Namioka [74]: if X is a complete metric space (more generally Čech-complete or strongly countably complete), K is compact and $f: X \times K \rightarrow \mathbb{R}$ is separately continuous, then there is a dense G_δ subset $D \subseteq X$ such that f is jointly continuous at each point of $D \times K$. The theorem showed that the correct hypotheses are completeness-type on one factor and compactness-type on the other, and introduced fragmentability and the quasi-continuity of the induced map $X \rightarrow C(K)$. Christensen [75] replaced metric completeness by weaker Baire-type conditions and linked the problem to Souslin and K -analytic spaces and to selection theorems. Talagrand [76] studied Namioka spaces and co-Namioka compact spaces, connecting them with Radon–Nikodým and Corson compacta; Bouziad [77] proved stability of co-Namioka compacta under products. Piotrowski's surveys [78] remain the most convenient entry point.

A unifying observation, already visible in Namioka [74] and developed by Mibu [79] and the Slovak school [64], is that separate continuity typically yields quasi-continuity (equivalently, Levine's semi-continuity [27]) of the associated map into a function space, and that quasi-continuity plus one further condition (continuity in the second variable, sub-continuity, or pointwise-to-norm fragmentability) yields joint continuity on a dense set. Thus the “semi” continuities of general topology are exactly the continuities that arise naturally in the separate-to-joint problem. A parallel strand concerns separately continuous algebraic operations: Ellis [80] proved that a locally compact Hausdorff group with separately continuous multiplication and continuous inverse is a topological group, a result of the same “weak continuity plus compactness forces continuity” type as in Section 2.2.

7. Uniform continuity

7.1 Classical theory and UC spaces

Uniform continuity for metric spaces — for each $\varepsilon > 0$ there is $\delta > 0$ with $d_Y(f(x), f(x')) < \varepsilon$ whenever $d_X(x, x') < \delta$ — was isolated by Heine [81], who proved that a continuous function on a closed bounded interval is uniformly continuous. Uniform continuity implies continuity but not conversely, is preserved by composition, and is *not* a topological property, since it depends on the metric (the homeomorphism $x \mapsto x^3$ of $\mathbb{R}$ and its inverse give the standard example). Uniformly continuous maps preserve Cauchy sequences, total boundedness and uniform convergence, but not completeness or boundedness.

Atsugi [82] answered the question of which metric spaces have the property that every continuous real-valued function is uniformly continuous; these are now called Atsugi or UC spaces. Equivalent characterisations include: (i) every continuous $f: (X, d) \rightarrow \mathbb{R}$ is uniformly continuous; (ii) the set X' of accumulation points is compact and, for every $\varepsilon > 0$, the set $\{x: d(x, X') \geq \varepsilon\}$ is uniformly discrete; (iii) every open cover has a Lebesgue-number refinement; (iv) every sequence (x_n) with $d(x_n, X \setminus \{x_n\}) \rightarrow 0$ has a cluster point. Hueber [83] and Beer [84] gave streamlined treatments and refinements, including characterisations through the distance functional and hyperspace topologies. The gap between continuity and uniform continuity is thereby governed by a local-compactness-plus-uniform-discreteness dichotomy.

7.2 Intermediate classes and uniform structures

Between continuity and uniform continuity the literature places the hierarchy

$$\text{Lipschitz} \Rightarrow \text{Hölder} \Rightarrow \text{uniformly continuous} \Rightarrow \text{Cauchy-continuous} \Rightarrow \text{continuous},$$

together with the incomparable classes of uniformly locally and locally uniformly continuous maps. Snipes [85] isolated the Cauchy-continuous maps, proving that they are exactly the maps extending continuously to the completion — the metric prototype of an extension theorem driven by a weak continuity. Weil [86] axiomatised uniform spaces by a filter of entourages, Bourbaki [10] developed the theory systematically, and Isbell [87] gave the definitive monograph treatment. The parallel theory of proximity spaces was introduced by Efremovič [88] and expounded with Smirnov's compactification theory by Naimpally and Warrack [89]; proximity-continuous maps form a further intermediate class. Two facts are relevant for comparison: a space is uniformisable if and only if it is completely regular, and on a compact Hausdorff space there is a unique uniformity, whence every continuous map from a compact Hausdorff space to a uniform space is uniformly continuous — the general form of Heine's theorem.

8. Compactness, connectedness, convergence and function spaces

8.1 Compactness and connectedness

Alexandroff and Urysohn [13] established bicomcompactness as the central finiteness notion, Tychonoff [16] proved its productivity, and his second paper [90] introduced the product topology on function spaces. Covering generalisations — Lindelöf, countably compact, sequentially compact, pseudocompact, paracompact and metacompact spaces — are structured by A. H. Stone's theorem that metrizable spaces are paracompact [91] and Michael's refinement characterisations [92]; in metric spaces compactness, countable compactness, sequential compactness and completeness plus total boundedness coincide, whereas in general spaces they are pairwise distinct [5]. Continuous surjections preserve compactness, countable compactness, Lindelöfness and pseudocompactness but not paracompactness or local compactness; perfect maps preserve and reflect a much larger list. Arhangel'skii's memoir [93] and Michael's "quintuple quotient quest" [94] are the standard references for the systematic study of classes of maps (open, closed, perfect, quotient, bi- and tri-quotient). The contrast with the generalised theory is instructive: the classical theory is organised by classes of maps closed under composition, which the generalised theory lacks.

Connectedness, in Kuratowski's formulation [9] and Whyburn's development [95], is the most robust property under weakening of continuity: nearly every generalised continuity preserves it, because the proof requires only the absence of a nontrivial partition into sets of the relevant type. For metric spaces the Hahn–Mazurkiewicz theorem characterises Peano continua as the continuous images of the unit interval $[0,1]$, the sharpest statement of what continuity can do to connectedness.

8.2 Convergence

Three strands of the convergence literature are relevant. The *net/filter* strand [19], [21], [20] supplies the characterisations of continuity, compactness and completeness. The *sequential* strand, initiated for general spaces by Franklin [96], identifies the sequential and Fréchet–Urysohn spaces in which sequences suffice to determine topology and continuity; this class includes all metric spaces. The *axiomatic convergence* strand, culminating in Dolecki and Mynard [6], develops topology directly from convergence relations, with continuity defined by $f(\lim \mathcal{F}) \subseteq \lim f(\mathcal{F})$; here the Čech closure operators associated with generalised closed sets [44] appear as precisely the pretopological operators.

8.3 Function spaces and hyperspaces

The study of spaces of continuous functions begins with the Ascoli–Arzelà theorems [97], [98], which characterise relative compactness in $C(K)$ by pointwise boundedness and equicontinuity. Fox [99] posed the problem of topologies for which the exponential correspondence $C(X \times Y, Z) \cong C(X, C(Y, Z))$ is a bijection; Arens [100] and Arens and Dugundji [101] developed admissible and proper topologies, with the compact-open topology canonical when Y is locally compact Hausdorff, and Day and Kelly [102] identified core-compactness as the correct condition for exponentiable spaces, explaining the failure of **Top** to be cartesian closed. The separate-to-joint problem of Section 6 is the exponential-law problem read with the topology of pointwise convergence. The comparison of function-space topologies is developed by Kelley [1] and McCoy and Ntantu [103], and the duality programme in which properties of $C_p(X)$ reflect those of X by

Arhangel'skii [104]. Michael [105] and Beer [106] founded and consolidated the theory of hyperspace topologies; semi-continuity of set-valued maps in the analytic sense is continuity for suitable hyperspace topologies, and selection theorems are the technical engine behind Christensen's work [75]. Finally, the uniformity of uniform convergence on Y^X [10], [87] yields the theorems that uniform limits of continuous maps are continuous, that $C(X, Y)$ is closed in Y^X , and that completeness of Y implies that of $C(X, Y)$; Kelley and Namioka [107] develop the linear-topological consequences.

9. Separation axioms and metrization

The axioms T_0 , T_1 , T_2 (Hausdorff [8]), T_3 , $T_{3\frac{1}{2}}$ (Tychonoff [16]), T_4 and T_5 , together with their non-implications [5], form the classical ladder. Their interaction with continuity operates through three mechanisms: Hausdorffness of the codomain makes graphs and coincidence sets closed and limits unique; complete regularity of the domain makes $C(X, \mathbb{R})$ large enough to separate points from closed sets [14], [16]; and normality makes extension possible [15]. The theorems of Urysohn and Tietze are thus not merely results about separation but the principal technique by which continuous functions are *constructed*.

The metrization problem was initiated by Fréchet [7] and advanced by Urysohn [14], whose theorem (a second countable regular T_1 space is metrizable) is the first general result. Moore [108] developed the axiomatic approach leading to Moore spaces; the complete solution was obtained independently by Nagata [109], Smirnov [110] and Bing [111]: a T_1 space is metrizable if and only if it is regular and has a σ -locally finite base. The proofs rest on Stone's theorem [91] and Michael's refinement theory [92]; Frink [112] gave a neighbourhood-based criterion that does not presuppose paracompactness. The monograph of Arhangel'skii and Ponomarev [113] and Gruenhage's survey [114] document generalised metric spaces and the role of mapping theorems in that theory — the classical precedent for asking which generalised continuities preserve or reflect metrizability-type properties.

A distinctive feature of the generalised-continuity literature is its concentration on axioms below T_1 : Levine's $T_{1/2}$ [43], Dunham's singleton characterisation [44], the semi-axioms [61], [62], [65], the analysis of Jankovic and Reilly [63], the $T_{3/4}$ axiom [49] and the pre- $T_{1/2}$ phenomenon [69] form a dense network of axioms, each defined by the collapse of a class of generalised closed sets into closed sets. The characteristic theorem has the form

$$X \text{ satisfies axiom } \square \iff \text{every } \square_1\text{-closed set is } \square_2\text{-closed} \iff \text{every singleton is of type } \square,$$

with the characteristic application

$$X \text{ satisfies } \square \iff \text{every } \square_1\text{-continuous map out of (or into) } X \text{ is } \square_2\text{-continuous.}$$

These “collapse” theorems are the most satisfying results in the generalised-continuity literature, since they convert a comparison of map classes into an intrinsic property of a space.

10. Critical analysis and open problems

A reading of the literature surveyed above suggests the following observations.

Proliferation without organisation. The number of named classes of generalised open sets and continuities now runs into the hundreds, and many are provably equivalent but introduced independently under different names. Quasi-continuity and semi-continuity [27], [64] are the best-known example; the pairs sg -/ gs - [42], [65] and ag -/ ga - [45], [46] differ only in the placement of a quantifier. The absence of a standard nomenclature makes duplication endemic and the literature difficult to audit.

Structural deficiency. The families $SO(X)$, $PO(X)$, $\beta O(X)$ and $bo(X)$ are closed under unions but not intersections, and the families of g -, sg -, gs -, ag - and rg -closed sets are closed under neither. Consequently the closure operators are not additive, the continuities are not closed under composition [56], [58], [41], and the classes possess no initial or final structures, no canonical products or quotients and no exponential objects. The usual repair, an auxiliary irresolute class, is local and does not yield a theory.

Repetition of structural results. The frameworks of Maki [51], Popa and Noiri [52] and Császár [53] show that most published characterisation, decomposition and preservation theorems about particular generalised continuities follow from the axioms of minimal structures or generalised topologies alone. Special-case proofs nevertheless continue to appear, and the frameworks have not been used systematically to delimit which results depend on the specific interpolation inequality.

Weak coupling to metric phenomena. The generalised literature is almost entirely point-set topological. Moduli of continuity, uniform versions of semi- and sg-continuity, Lipschitz or Cauchy analogues and behaviour under completion are rarely studied, and the metric literature on UC spaces [82], [83], [84] and Cauchy-continuity [85] has not been connected to the generalised classes.

Asymmetry and isolation. Most papers prove preservation theorems; few prove reflection theorems of the Arhangel'skii–Michael type [93], [94]. Counterexamples are drawn mostly from small finite spaces rather than familiar infinite ones such as the Sorgenfrey line or the Niemytzki plane [5]. The separate-to-joint literature, whose central tool is quasi-continuity [64], [74], [78], and the general-topological literature on semi-continuity cite one another very little. Finally, the function-space side [99], [100], [101], [102], [103], [104] is neglected: the sets of semi- and sg-continuous maps are not closed under pointwise limits, their natural topologies have not been determined and no Ascoli-type theorem is available.

These observations yield eight open problems, collected in Table 2.

Table 2. Open problems arising from the review.

Problem	Question	Related work
1. Unified framework	Is there a single operator-theoretic definition of which classical, semi-, sg-, gs- and uniform continuity are all instances, extending minimal structures to uniform and metric notions?	[52], [53], [51], [54]
2. Composability	What is the largest class of maps with which a given generalised continuity composes on each side, and under what hypotheses is composition automatic?	[56], [58], [41], [66]
3. Uniform analogues	Do uniform semi- and sg-continuity admit an Atsugi-type theorem: for which metric spaces is every semi-continuous real function uniformly semi-continuous, or continuous?	[82], [85]
4. Reflection theorems	Which generalised continuities reflect metrizable, first countability or paracompactness under closedness-type hypotheses?	[93], [94], [109]
5. Compactness notions	Can semi-compact, S-closed, quasi-H-closed, nearly and almost compact be organised by a single principle, and related to sg-type continuities?	[60], [59]
6. Separate weak continuity	Do Namioka- and Ellis-type theorems persist for separately semi- or sg-continuous functions, and what is the correct conclusion?	[74], [80]
7. Function spaces	What are the topological and uniform structure, closedness properties and Ascoli-type criteria for the spaces of semi- and sg-continuous maps?	[97], [98]
8. Collapse diagram	Can the diagram of $T_{1/2}$ -type axioms and the map-class collapses they characterise be completed, and which are automatic in metric spaces?	[43], [44], [49], [65], [69]

11. Conclusions and directions for future work

The literature on continuous mappings divides into a mature classical theory, organised by classes of maps closed under composition and by tight correspondences between properties of spaces and function spaces, and a very large but loosely organised body of work on generalised continuities. The latter has produced many characterisation, decomposition and preservation theorems, and the “collapse” theorems of the $T_{1/2}$ family show the form that genuinely structural results can take. Its weaknesses — proliferation, lack of composability, repetition, and isolation from metric, uniform and function-space theory — are, however, systematic rather than accidental.

Three directions for future work follow from the review. First, generalised continuity should be studied *comparatively and structurally* within operator-based frameworks in the spirit of Maki, Popa–Noiri and Császár, extended upward to uniform and metric structures and sideways to the separate-to-joint and function-space problems, with the framework used as an audit tool to separate structural results from special ones. Second, every new class or theorem should state what fails without each hypothesis, with counterexamples from standard infinite spaces, and metric specialisations should be stated in parallel

with general-topological results. Third, the link between quasi-continuity and the Namioka-type theorems offers the most promising bridge between two literatures that have developed in isolation. Fuzzy, soft, bitopological and ideal-topological generalisations lie outside the scope of this review; a framework of the kind proposed should, however, be formulated so as to admit them.

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