

Iterative and Semi-Analytical Methods for Nonlinear Differential Equations: A Review with Emphasis on the Daftardar-Gejji–Jafari Method

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Abstract. Nonlinear ordinary and partial differential equations rarely admit closed-form solutions, and their study has therefore passed from analytical solvability to approximation with controlled error. This article reviews the principal families of methods for nonlinear differential equations: classical root-finders and time integrators, iterative and semi-analytical schemes (Picard, Adomian decomposition, homotopy perturbation, variational iteration and homotopy analysis), and discretisation-based PDE solvers (finite difference, finite element, finite volume, spectral and multigrid). Particular attention is given to the Daftardar-Gejji–Jafari method (DJM), also called the New Iterative Method, which generates a convergent series without Adomian polynomials, embedding parameters or Lagrange multipliers. We survey its formulation, convergence theory and applications, the modified and hybrid schemes built around it, and the treatment of three benchmark problems (the Fisher, Newell–Whitehead–Segel and Boussinesq equations) and of stiff and chaotic systems. A comparative analysis shows that discretisation methods are theoretically mature but costly or geometrically restrictive, whereas semi-analytical series methods give closed-form solutions at low algebraic cost but rest on a thinner theory and have a limited domain of convergence. Five research gaps are identified.

Keywords: *nonlinear differential equations; Daftardar-Gejji–Jafari method; new iterative method; Adomian decomposition; homotopy methods; Fisher equation; Newell–Whitehead–Segel equation; Boussinesq equations; stiff systems; chaotic systems*

1. Introduction

Nonlinear ordinary and partial differential equations (ODEs and PDEs) are the mathematical backbone of applied science and engineering. Population dynamics, reaction–diffusion processes, hydrodynamic wave propagation, combustion, convection instabilities, chemical kinetics and control systems are all governed by equations in which the unknown function appears nonlinearly. Unlike their linear counterparts, such equations rarely admit closed-form solutions, and the classical arsenal of superposition, separation of variables, integral transforms and eigenfunction expansion fails in all but a handful of integrable cases [1], [2]. Analytical solvability has consequently been replaced by approximation with controlled error, through numerical, semi-analytical and iterative methods whose convergence and stability can be rigorously characterised [3].

This article reviews that literature along a deliberate progression. Section 2 covers the classical foundations, namely root-finding schemes and single- and multi-step time integrators. Section 3 treats iterative and semi-analytical methods for nonlinear ODEs, from Picard’s successive approximation to the decomposition, homotopy and variational families. Section 4 summarises discretisation-based PDE solvers. Section 5 is devoted to the Daftardar-Gejji–Jafari method (DJM), also designated the New Iterative Method (NIM), introduced in 2006; Section 6 reviews modified and hybrid schemes. Section 7 examines the Fisher, Newell–Whitehead–Segel (NWS) and Boussinesq equations as benchmark problems, and Section 8 the specialised literature on stiff and chaotic systems. Section 9 consolidates what is known about convergence, stability and error analysis, and Section 10 gives a comparative synthesis and identifies research gaps.

Three observations recur as critical threads. First, semi-analytical series methods frequently show excellent accuracy on small domains but degrade rapidly as the independent variable grows, an issue only partly addressed by multistage and Padé-based remedies. Second, rigorous convergence proofs lag well behind applications: many published papers verify accuracy numerically against a known exact solution rather than establishing error bounds a priori. Third, comparative studies are often methodologically inconsistent, with different truncation orders, error norms and test problems, which makes genuine method ranking difficult.

2. Classical methods

2.1 Root-finding schemes

The Newton–Raphson iteration $x_{n+1} = x_n - f(x_n)/f'(x_n)$ is the archetype of iterative methods for nonlinear problems and the conceptual ancestor of DJM. Traub [4] established the systematic theory of one-point and multipoint iteration functions and Ostrowski [5] the foundational convergence analysis; the method converges locally and quadratically when f' does not vanish at the root and f is twice

continuously differentiable [6]. Its generalisation to operator equations $F(x) = 0$ in Banach spaces, the Newton–Kantorovich theorem, matters here because nonlinear differential equations recast in integral form are exactly such equations [3]. Deuffhard [7] documented the affine-invariant theory and damped and inexact variants, and Kelley [8] Newton–Krylov and quasi-Newton implementations for the large sparse Jacobians arising from PDE discretisation. Recorded limitations are dependence on a good initial guess, the cost of forming and factorising Jacobians, and order degradation at multiple roots; Kung and Traub [9] related the number of function evaluations to the attainable order, which has driven the proliferation of higher-order and derivative-free variants.

The bisection method, resting on the intermediate value theorem, is the canonical globally convergent but slowly (linearly, with rate $1/2$) converging bracketing scheme [6]; hybrid root-finders such as Dekker’s and Brent’s methods combine its guarantees with the speed of secant or inverse quadratic interpolation. In differential equations it appears chiefly inside shooting methods for boundary value problems. The secant method replaces the derivative by a finite difference of two previous iterates, giving order equal to the golden ratio $\phi \approx 1.618$ without requiring f' [4], [5]; it is the prototype of the quasi-Newton class, including Broyden’s method, and per function evaluation it can be competitive with Newton’s method [6], [9].

2.2 Time integrators for ODEs

The explicit Euler method is the simplest consistent one-step integrator for $y' = f(t, y)$; its first-order accuracy and restrictive stability region are treated by Butcher [10] and by Hairer, Nørsett and Wanner [11]. The implicit Euler method is A- and L-stable, making it the simplest viable scheme for stiff problems at the cost of a nonlinear algebraic solve per step, and the trapezoidal rule extends this to second order [12]. Runge–Kutta (RK) methods reach higher order by evaluating f at several stages within a step [10], [11]; the classical fourth-order scheme remains the default for smooth non-stiff problems, and embedded pairs supplying local error estimates for adaptive step control were introduced by Fehlberg [13] and refined by Dormand and Prince [14]. Explicit RK methods have bounded stability regions, order barriers limit efficiency at high order, and they do not preserve geometric invariants such as symplectic structure, which matters for conservative and chaotic problems; implicit families (Gauss–Legendre, Radau IIA, SDIRK) overcome the stability restriction.

Linear multistep methods use several previous steps to reach high order cheaply. The Adams–Bashforth–Moulton predictor–corrector pairs suit non-stiff problems, while the backward differentiation formulae (BDF), systematised by Gear [15], became the standard tool for stiff systems. Dahlquist’s results [16] frame the theory: consistency plus zero-stability implies convergence, and no A-stable linear multistep method can exceed order two, so BDF methods of orders 1–6 are only $A(\alpha)$ -stable, a compromise governing the design of modern stiff solvers [12], [11].

3. Iterative and semi-analytical methods for nonlinear ODEs

3.1 Picard and Newton-type iteration

Picard’s method of successive approximations, formalised in the Picard–Lindelöf theorem, converts the initial value problem into the Volterra equation $y(t) = y_0 + \int f(s, y(s)) ds$ and iterates the right-hand side [17]; convergence follows from the Banach contraction principle under a Lipschitz condition. DJM can be interpreted as a structurally generalised and accelerated Picard-type iteration acting on a decomposed nonlinear operator. The literature identifies two weaknesses: successive quadratures quickly become intractable for non-polynomial nonlinearities, and convergence is linear and only local. Rach [18] compared Picard iteration with the Adomian decomposition method (ADM), showing that the two generate different series and that ADM generally needs fewer terms for comparable accuracy.

Newton’s method enters differential equations in three guises: quasilinearisation, in which the operator is linearised about the current iterate to give a sequence of linear boundary value problems converging quadratically [3], [7]; implicit time stepping, in which Newton iteration solves stage equations with Jacobian reuse and Newton–Krylov strategies [8], [12]; and nonlinear multigrid. Ortega and Rheinboldt [3] remain the standard reference for the convergence theory. Quadratic convergence is purchased with derivative information and sensitivity to the initial guess, which is the cost decomposition-, homotopy-type methods and DJM seek to avoid.

3.2 Adomian decomposition

Adomian’s method [19] decomposes the equation into linear and nonlinear parts, inverts the linear operator and expands the nonlinearity in Adomian polynomials A_n depending on $u_0, \dots, u_n$, giving a rapidly convergent series without linearisation, perturbation or discretisation. Convergence theory was developed by Cherruault and co-workers [20], [21], [22] in a functional-analytic setting. Three limitations persist: the polynomials become algebraically formidable for strong or non-polynomial nonlinearities; the truncated series has a finite effective radius of convergence; and the choice of decomposition and zeroth component materially affects performance. Wazwaz’s reliable modification [23] and the Wazwaz–El-Sayed modification [24] redistribute the source term across components and demonstrably accelerate convergence. The cost of Adomian polynomials is precisely what DJM was designed to eliminate.

3.3 Homotopy and variational methods

He's homotopy perturbation method (HPM) [25], [26] embeds a linear problem with known solution into the target problem through a parameter $p \in [0,1]$ and expands in powers of p ; unlike classical perturbation theory it needs no small physical parameter. Ghorbani [27] introduced "He polynomials", showing that the HPM expansion of the nonlinearity can be organised in a form comparable to, and easier to compute than, Adomian polynomials. Critical appraisals note that accuracy depends strongly on the operator splitting and initial approximation, that the convergence domain is local, and that in many examples HPM reduces to a rearrangement of a Taylor-type series [28], [29].

He's variational iteration method (VIM) [30] builds a correction functional with a Lagrange multiplier λ determined variationally, $u_{n+1}(t) = u_n(t) + \int \lambda(s)\{Lu_n + N\tilde{u}_n - g\}ds$, avoiding both discretisation and Adomian polynomials; when λ is correctly identified it often converges faster than Picard iteration, of which it is a weighted generalisation. He and Wu [31] surveyed its developments, and Odibat [32] gave sufficient convergence conditions and error estimates by treating VIM as a fixed-point iteration in a Banach space. Documented limitations are the dependence on correct multiplier identification, difficulties for strongly nonlinear or highly oscillatory problems and, as with ADM and HPM, deterioration over extended domains.

Liao's homotopy analysis method (HAM) [33] introduces an auxiliary linear operator, an auxiliary function and a convergence-control parameter $\hbar$; HPM is recovered as the special case $\hbar = -1$. Because the series is parameterised, its region and rate of convergence can be adjusted rather than merely observed, making HAM the method of choice for strongly nonlinear problems and unbounded domains. Liao's optimal HAM [34] and the Optimal Homotopy Asymptotic Method of Marinca and Herisanu [35] determine the control constants by residual minimisation. Liang and Jeffrey [28] compared HAM and HPM on an evolution equation and concluded that HPM's convergence is not guaranteed in general whereas HAM's parameter can enforce it. The counterweight is the additional effort of optimising auxiliary parameters and the algebraic complexity of the deformation equations.

4. Discretisation methods for nonlinear PDEs

Discretisation-based solvers remain the industrial standard for large-scale computation. *Finite difference methods* replace derivatives by difference quotients on a structured grid; LeVeque [36] gives the modern treatment, and for nonlinear parabolic problems the Crank–Nicolson scheme [37] offers second-order accuracy with unconditional linear stability, at the price of a nonlinear algebraic system per level, handled by Newton or Picard iteration or by linearised and semi-implicit variants. Stability follows von Neumann Fourier analysis or matrix/energy methods [38], and for hyperbolic problems the CFL condition [39] constrains explicit schemes. Disadvantages are geometric inflexibility and, for conservation laws, the need for specialised flux treatment.

Finite element methods, based on weak formulations and piecewise polynomials on unstructured meshes, offer geometric flexibility and a functional-analytic error theory in Sobolev norms [40], [41]; nonlinear systems are solved by Newton–Raphson or arc-length continuation, convection-dominated problems need stabilised or discontinuous Galerkin formulations, and adaptive refinement resolves travelling fronts of the kind exhibited by Fisher and NWS solutions [42]. *Finite volume methods* integrate over control volumes so that conservation holds discretely, the decisive property for nonlinear conservation laws [43]; high-resolution Godunov-type schemes with Riemann solvers and slope limiters capture shocks without spurious oscillations [44], though limiting can smear sharp fronts. *Spectral methods* expand the solution in globally smooth bases (Fourier, Chebyshev, Legendre) and converge exponentially for analytic solutions [45], [46]; they are the reference standard against which semi-analytical approximations, DJM included, are frequently validated, but need simple geometry, produce stiff dense operators (Chebyshev diffusion eigenvalues scale like N^2) and lose accuracy at discontinuities (Gibbs phenomenon). *Multigrid methods* give mesh-independent convergence by combining relaxation on a hierarchy of grids [47], [48], [49]; for nonlinear problems the Full Approximation Scheme or Newton–multigrid is used [48], [50], with Krylov solvers and preconditioning completing the machinery for very large sparse nonlinear systems [50].

5. The Daftardar-Gejji–Jafari method

5.1 Origin and formulation

DJM was introduced in 2006 as "An iterative method for solving nonlinear functional equations" [51]. Its stated motivation was comparative: ADM requires the laborious construction of Adomian polynomials, HPM the construction of a homotopy and expansion in an artificial parameter, and VIM the identification of a Lagrange multiplier; DJM dispenses with all three. The method is therefore referred to both as DJM and as the New Iterative Method (NIM), the designations being used interchangeably [52], [53].

DJM addresses the nonlinear functional equation

$$u = f + N(u)$$

where N is a nonlinear operator on a Banach space and f is a known function — the form obtained whenever a differential equation is inverted against its highest-order linear operator. The solution is sought as a series $u = \sum u_i$, and the nonlinear operator is decomposed as

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty} \{ N(\sum_{j=0}^i u_j) - N(\sum_{j=0}^{i-1} u_j) \}$$

which yields the simple recursion

$$u_0 = f, \quad u_1 = N(u_0), \quad u_{m+1} = N(u_0 + \dots + u_m) - N(u_0 + \dots + u_{m-1}), \quad m \geq 1,$$

so that the m -term truncation satisfies the original equation to the corresponding order [51]. The decisive structural feature is that each new term requires only two evaluations of the nonlinear operator on partial sums: no differentiation of the nonlinearity, no polynomial recursion, no auxiliary parameter and no multiplier.

Bhalekar and Daftardar-Gejji [54] established the convergence theory, proving that the series converges absolutely under conditions on the Fréchet derivatives of N (boundedness of $\|N^{(n)}(u_0)\|$ and of the terms), with explicit sufficient conditions and error estimates. They also clarified the relation to Adomian's method, showing that the two series coincide under certain conditions while DJM's terms require substantially less algebraic labour.

5.2 Applications

For ODEs, the originating paper [51] treated nonlinear functional and integral equations. Daftardar-Gejji and Bhalekar extended the method to fractional-order equations, where the Riemann–Liouville or Caputo integral plays the role of the inverted linear operator, obtaining closed-form series for fractional diffusion-wave problems [52]; Hemeda [53] applied it to n th-order integro-differential equations, reporting accuracy comparable or superior to ADM and VIM with fewer steps; and Jafari and co-workers treated systems of fractional PDEs [55]. A recurrent finding is that for polynomial nonlinearity the DJM series reproduces the Taylor expansion of the exact solution, so exact solutions are recovered when the solution is polynomial or the series sums in closed form; equally recurrent is the loss of accuracy for large t .

For PDEs, Bhalekar and Daftardar-Gejji [56] solved Burgers-type and reaction–diffusion problems and compared the results with ADM and exact solutions. The operator N incorporates spatial derivatives and inversion is performed in time, so each iterate requires only differentiation of previously computed analytic expressions, a practical advantage over discretisation: no mesh, no stability restriction and no linear solver. Studies followed on fractional diffusion-wave equations [52], systems of fractional PDEs [55] and the NWS equation [57]. Related iterative frameworks developed independently include the semi-analytical technique of Temimi and Ansari [58], applied by Al-Jawary [59] to thin-film flow, which situate DJM within a family of operator-level iterative methods rather than as an isolated technique.

5.3 Advantages and limitations

The literature converges on a consistent appraisal. *Advantages*: (i) no Adomian polynomials, homotopy parameter or Lagrange multiplier, so the recursion is elementary and easily implemented in a symbolic algebra system [51], [53]; (ii) two evaluations of N per term, a lower complexity per term than ADM for strong nonlinearities; (iii) applicability to integer- and fractional-order ODEs, PDEs and integro-differential equations [52], [55]; (iv) closed analytic solutions permitting differentiation and parameter studies without interpolation; (v) available convergence conditions and error bounds [54].

Limitations: (i) convergence is conditional on boundedness of the Fréchet derivatives and the practical radius of convergence may be small, so accuracy deteriorates on large domains, the most frequently reported drawback; (ii) performance depends on the splitting of the equation and the choice of f , for which no optimality theory exists; (iii) the higher iterates suffer “expression swell”, limiting the number of terms; (iv) the method does not itself address stiffness or long-time chaotic dynamics, where qualitative fidelity rather than truncation error is the criterion; (v) comparative claims often rest on few terms and a single test problem, with little systematic benchmarking against high-resolution numerical solutions.

6. Modified and hybrid iterative methods

A substantial literature modifies and hybridises semi-analytical methods, driven by the shared limitation of a locally convergent series whose accuracy decays over extended domains. In the decomposition family, Wazwaz's modifications [23], [24] redistribute the source term between components, sometimes yielding the exact solution in one iteration; Ghorbani and Saberi-Nadjafi [60] and Ghorbani [27] replaced Adomian polynomials by He polynomials; and Hosseini and Nasabzadeh [61] examined convergence acceleration and the role of the initial component. In the homotopy and variational families, Noor and Mohyud-Din [62] developed modified VIM schemes coupled with He's polynomials for higher-order boundary value problems, Odibat and Momani [63] restructured the HPM expansion to enlarge the convergence region (demonstrated on the fractional Riccati equation), and the optimal HAM [34] and OHAM [35] systematise the optimisation of convergence-control parameters.

The most effective general remedy for the finite convergence radius is the *multistage* approach: the domain is partitioned into subintervals and the series is restarted on each, with the terminal value of one stage supplying the initial condition of the next. Alomari, Noorani and Nazar [64] implemented a multistage HAM for the Chen system and obtained accurate long-time trajectories where single-stage HAM failed; analogous formulations exist for ADM, VIM, HPM and the differential transform method [65], and the same principle

transfers directly to DJM. Coupling with the Laplace transform linearises the inversion step and can extend the domain of validity, while Padé approximants convert a truncated power series into a rational function with far better large-argument behaviour and the ability to represent boundary-layer structure. The reduced differential transform method of Keskin and Oturanç [66] offers a further iterative alternative for nonlinear PDEs. Hybrids with classical discretisation, semi-analytical in time and spectral or finite-difference in space, form a smaller but growing literature in which comparative evidence remains thin.

7. Benchmark equations

7.1 Fisher's equation

Fisher's equation, $u_t = D u_{xx} + \alpha u(1 - u)$, was introduced by R. A. Fisher in 1937 to model the spread of an advantageous gene [67] and analysed in greater generality in the same year by Kolmogorov, Petrovskii and Piskunov [68] (hence Fisher–KPP). It is the canonical reaction–diffusion equation, combining Fickian diffusion with logistic growth, and admits travelling fronts connecting $u = 0$ to $u = 1$ for all speeds $c \geq 2\sqrt{\alpha D}$. Aronson and Weinberger [69] established the rigorous theory of front propagation and the selection of the minimal wave speed from compactly supported data, the benchmark against which numerical and semi-analytical solutions are judged.

Gazdag and Canosa [70] gave an early pseudospectral study; finite-difference work concentrated on schemes preserving positivity and boundedness of u , since the logistic term is physically meaningful only for $0 \leq u \leq 1$, and on compact high-order formulations [71]. Tang and Weber [42] used a Petrov–Galerkin finite element method on adaptive meshes, and Mittal and Jain [72] cubic B-spline collocation. Semi-analytically, Wazwaz and Gorguis [73] studied Fisher and generalised Fisher equations by ADM, Öziş and Koroğlu [74] obtained exact solitary-wave solutions by the Exp-function method, and Ağırseven and Öziş [75] applied HPM, which reproduces the leading front behaviour with few terms. Two findings recur. All single-stage series methods (ADM, HPM, VIM, HAM and DJM) are very accurate for small t but lose accuracy as the front propagates, because a truncated power series in t cannot represent a uniformly translating front over long times; and accuracy degrades as the reaction rate α grows relative to diffusion, since the effective time scale shortens. Multistage restarting and Padé resummation are the reported remedies.

7.2 The Newell–Whitehead–Segel equation

The NWS equation, $u_t = k u_{xx} + a u - b u^p$, is an amplitude equation governing slow modulation of convection rolls near the onset of Rayleigh–Bénard instability, derived independently by Newell and Whitehead [76] and by Segel [77]. It couples a linear instability (au), nonlinear saturation ($-bu^p$) and diffusive coupling, and for $p = 2$ coincides structurally with Fisher's equation. Because exact solutions exist for particular parameters, it is a standard test problem: Nourazar, Soori and Nazari-Golshan [78] showed that the HPM series sums to the known closed form, Aasaraai [79] applied the differential transform method, and Prakash and Kumar [80] applied VIM. Of particular significance for DJM, Patade and Bhalekar [57] derived approximate analytical solutions with the New Iterative Method, showing rapid convergence to the known exact solutions and favourable comparison with ADM and HPM. Classical schemes parallel those for Fisher's equation, with the added need to handle a general exponent p , which precludes simple quadratic linearisation. A methodological weakness is evident: many papers show agreement with an exact solution for small t and a single parameter set, without examining parameter sensitivity (large a , large p , small k), error over extended time intervals, or comparison with a high-resolution reference solution when no closed form exists.

7.3 The Boussinesq equations

Boussinesq's 1872 work on long waves in shallow water [81] produced the first weakly nonlinear, weakly dispersive wave model, retaining dispersion at leading order and thereby admitting solitary waves that the shallow-water equations cannot support. The family includes the “good” and “bad” Boussinesq equations $u_{tt} = u_{xx} \pm u_{xxx} + (u^2)_{xx}$, coupled systems for elevation and velocity, and the Boussinesq approximation in buoyancy-driven convection. Peregrine [82] derived the standard formulation for long waves on a beach, Madsen and Sørensen [83] improved its dispersion characteristics over varying bathymetry, and Bona, Chen and Saut [84] gave a rigorous derivation and classification of Boussinesq-type systems, establishing well-posedness and linear stability; the “bad” equation is linearly ill-posed for short waves. Clarkson and Kruskal [85] introduced the direct method for similarity reductions, producing exact solutions that serve as numerical benchmarks.

Manoranjan, Mitchell and Morris [86] carried out an early systematic numerical study of the good Boussinesq equation, documenting solitary-wave interaction, blow-up and sensitivity of long-time computation to scheme design, and Bratsos [87] developed a second-order finite-difference scheme with stability and conservation analysis. The dominant concerns are the fourth-order spatial derivative, which causes severe stiffness and tight step restrictions for explicit schemes; preservation of mass, energy and momentum over long integrations; and resolution of soliton interaction without phase error. Fourier pseudospectral methods with exponential or implicit–explicit integration are reported as the most accurate, while finite volume methods dominate coastal-wave modelling for their conservation and shock handling [44], [83]. Semi-analytically, Wazwaz [88] used VIM to obtain rational and solitary-wave solutions including the cubic Boussinesq equation, and ADM, HPM, HAM and reduced differential transform methods have been applied with accurate short-time results. DJM has been applied to related nonlinear evolution and wave equations [56], but the literature on DJM for Boussinesq systems, particularly coupled systems and long-time soliton interaction, is comparatively sparse. Reported difficulties

include expression swell, loss of invariant conservation in truncated series and inability of a short series to represent soliton interaction; these equations test iterative methods far more severely than the scalar reaction–diffusion problems.

8. Stiff and chaotic systems

8.1 Stiff ODEs

Stiffness was identified by Curtiss and Hirschfelder [89] in chemical kinetics, where widely separated time scales force explicit integrators to take steps dictated by the fastest decaying transient long after it has died: a restriction imposed by stability, not accuracy. Hairer and Wanner [90] remains the definitive reference, with Shampine [91] and Ascher and Petzold [92] oriented towards computation and differential-algebraic problems. The theory rests on the test equation $y' = \lambda y$ and the notions of A-, L-, $A(\alpha)$ -stability and stiff decay and, for nonlinear problems, B-stability [90], [16]. Dahlquist's second barrier explains the architecture of modern stiff solvers: accept $A(\alpha)$ -stability with higher-order BDF, or use implicit Runge–Kutta methods (Radau IIA, SDIRK, Gauss) that can be high-order and L- or B-stable.

The principal families are BDF/Gear methods in production codes from DIFSUB [15] to Hindmarsh's ODEPACK/LSODE, which introduced automatic stiff/non-stiff switching [93]; implicit RK methods, especially Radau IIA [90]; Rosenbrock methods [94], [95], linearly implicit schemes that avoid nonlinear stage solves when the Jacobian is available; IMEX and operator-splitting methods that treat stiff diffusion implicitly and reaction explicitly, natural for the reaction–diffusion equations above; and exponential integrators, which integrate the stiff linear part exactly, introduced by Cox and Matthews [96] and implemented stably (ETDRK4) by Kassam and Trefethen [97] for fourth-order stiff PDEs such as Boussinesq-type problems. Shampine and Reichelt [98] documented the MATLAB ODE suite, including ode15s and ode23s, and Cash [99] reviewed BDF, implicit RK and extrapolation approaches; these provide reference implementations for assessing new schemes. The literature is clear that truncated power-series methods (ADM, HPM, VIM, HAM and DJM alike) are intrinsically ill-suited to stiff problems in single-stage form, because the stiff transient confines the useful interval to the order of the fastest time constant. Reported remedies are multistage restarting [64], Padé resummation, which can confer A-stability-like behaviour on the rational approximant, and hybridisation within an implicit framework.

8.2 Chaotic systems

Lorenz's 1963 paper [100] showed that low-dimensional deterministic ODEs can exhibit aperiodic, sensitively dependent dynamics, and the Lorenz system became the universal benchmark for integrating chaotic flows; Rössler [101], Chen and Ueta [102] and Sprott [103] broadened the standard test set. Sensitive dependence means any numerical trajectory diverges exponentially from the true one at a rate set by the largest Lyapunov exponent, so pointwise long-time accuracy is unattainable in principle and the criterion shifts to reproduction of invariant and statistical structure: attractor geometry, Lyapunov spectrum, fractal dimension and invariant measure. Benettin et al. [104] and Wolf et al. [105] provided the standard algorithms for Lyapunov exponents, the instruments for validating a chaotic integrator. Hairer, Lubich and Wanner [106] established geometric numerical integration, with symplectic, symmetric and volume-preserving methods that conserve qualitative structure over exponentially long times; for dissipative chaos the requirements are correct attractor reproduction and avoidance of spurious numerical attractors, and high-order adaptive RK schemes [14] and Taylor-series integrators are the standard tools.

Semi-analytically, Vadasz and Olek [107] used ADM to compute the transition to chaos in a Lorenz-type convection problem, showing that sufficiently small sub-intervals resolve it; Hashim and co-workers [108] reported that single-interval ADM fails over long times for hyperchaotic systems whereas a multistage implementation reproduces the attractor; and Alomari, Noorani and Nazar [64] reached the same conclusion for multistage HAM. The principle that emerges is that semi-analytical iterative methods, DJM included, become viable for chaotic systems only in multistage form, where they act as high-order one-step integrators with analytically derived coefficients. A parallel literature treats fractional-order chaotic systems: Diethelm, Ford and Freed [109] developed the standard predictor–corrector algorithm and Petráš [110] surveyed their modelling and simulation. Because DJM extends naturally to fractional operators [52], [55], this is a promising direction.

9. Convergence, stability and error analysis

For discretisation methods the theory is mature and unified. The Lax–Richtmyer equivalence theorem [111] states that for a consistent linear finite-difference scheme applied to a well-posed linear initial value problem, stability is necessary and sufficient for convergence. Stability is assessed by von Neumann analysis, eigenvalue analysis or energy methods [38], [36], with the CFL condition [39] restricting explicit hyperbolic schemes; for ODE integrators, convergence rests on consistency and zero-stability with Dahlquist's barriers delimiting what is achievable [16], [12]. Finite element convergence follows from the Lax–Milgram/Céa framework with a priori and a posteriori estimates [40], [41], finite volume convergence from discrete compactness and entropy arguments [43], [44], and spectral convergence from approximation theory [45], [46]. For nonlinear problems, however, these results are typically local or need structural assumptions (monotonicity, coercivity, entropy conditions), and global convergence for general nonlinear PDEs is open in most cases.

For semi-analytical methods the theory is markedly less complete, an asymmetry that is one of the central observations of this review. Cherruault [20] initiated the functional-analytic convergence analysis of ADM, and Abbaoui and Cherruault [22], [21] gave convergence conditions and error estimates, though results are local and the radius of convergence is rarely quantified for specific problems; Rach [18] clarified the relation to Picard iteration. Biazar and Ghazvini [29] established sufficient conditions for HPM, whereas Liang and Jeffrey [28] showed by counterexample that HPM convergence is not guaranteed and depends on the splitting. Odibat [32] treated VIM as a Banach-space fixed-point iteration; Liao [33] showed that h can be chosen to enforce HAM convergence, Turkyilmazoglu [112] analysed convergence acceleration and optimal h , and optimal HAM [34] converts this into residual minimisation. For DJM, Bhalekar and Daftardar-Gejji [54] proved convergence under conditions on the Fréchet derivatives, with error bounds.

Four shortcomings recur across this literature. (i) *Sufficient but not sharp conditions*: the theorems [54], [21], [32] are hard to verify for a given problem and give pessimistic bounds, so practical convergence far exceeds the guarantee. (ii) *Error measured against known exact solutions*: this validates an implementation but cannot establish reliability where no closed form exists, which requires residual-based or reference-solution-based estimates. (iii) *Absence of stability analysis*: unlike discretisation methods, semi-analytical methods are rarely analysed for stability at all, a serious omission for stiff and chaotic problems; multistage formulations, being effectively one-step integrators, do admit conventional stability-region analysis, an underexploited avenue [64], [108]. (iv) *Unreported computational cost*: comparisons usually report accuracy at a fixed number of terms without CPU time, symbolic expression size or memory.

10. Comparative analysis and research gaps

A number of studies have compared semi-analytical methods directly. Wazwaz [113] compared ADM with the Taylor series method, showing where the series coincide and where decomposition is advantageous. Momani and Odibat [114] compared HPM and VIM for fractional PDEs, finding both accurate, HPM simpler to implement and VIM faster to converge for some operator classes. Liang and Jeffrey [28] concluded that HPM is a special case of HAM whose convergence is not assured, and Ghorbani [27] showed that He polynomials can replace Adomian polynomials with less labour, effectively unifying the approaches. For DJM, the originating and follow-up papers [51], [54], [56], [53], [57] consistently report accuracy comparable to ADM, HPM and VIM at the same truncation order with markedly less algebraic effort. Table 1 summarises the methods reviewed.

Table 1. Comparative summary of the methods reviewed.

Method	Auxiliary objects required	Convergence control	Main limitation
Newton–Raphson	Derivatives / Jacobian	Local, quadratic	Initial guess; derivative cost [6], [7]
Secant	None (divided differences)	Local, order ϕ	No global guarantee [4], [5]
Runge–Kutta	Embedded pair for adaptivity	Order 4–8, adaptive	Explicit forms unstable for stiff problems [10], [14]
Multistep / BDF	Jacobian, starter	$A(\alpha)$ -stable, order ≤ 6	Dahlquist barriers [16], [15]
Picard	None	Local, linear	Intractable quadratures [17], [18]
ADM	Adomian polynomials	Conditional [21]	Polynomial cost; small radius [19], [23]
HPM	Homotopy construction	Not guaranteed [28]	Splitting-dependent [25], [26]
VIM	Lagrange multiplier	Conditional [32]	Multiplier identification [30]
HAM / OHAM	Operator L , function H , h	Adjustable, optimisable	Parameter-tuning cost [33], [34], [35]
DJM / NIM	None	Conditional, proved [54]	Small radius; expression swell [51]
FDM / FEM / FVM	Mesh, solver, limiters	Lax–Richtmyer; a priori / a posteriori; entropy [111], [40], [43]	Geometry; complexity; numerical diffusion [36]
Spectral	Smooth basis, transforms	Exponential for analytic data	Geometry; stiffness; Gibbs [45], [46]
Multigrid	Grid hierarchy, smoothers	Mesh-independent	Smoother design [47], [48]

The review supports five conclusions, collected in Table 2. Classical and discretisation-based solvers are theoretically mature but costly or geometrically restrictive, whereas semi-analytical series methods deliver closed-form solutions at low algebraic cost but rest on a thinner theory and are confined to limited domains of convergence. Within this family DJM is the most economical, yet it remains the least systematically benchmarked and the least developed in multistage, hybrid and stability-analysed form.

Table 2. Research gaps identified by the review.

Gap	Statement	Related work
1. Benchmarking of DJM	DJM has the most economical structure but is far less systematically benchmarked than ADM, HPM, VIM and HAM, especially for coupled and higher-order PDEs such as Boussinesq systems.	[51], [54], [56], [88]
2. Multistage DJM	The finite domain of convergence is the dominant limitation; multistage strategies are effective for ADM and HAM, but a multistage DJM with step-size control and stability characterisation is not established.	[64], [108]
3. Stability analysis	No published study subjects DJM to the stability-region and stiff-decay analysis routinely applied to RK and BDF methods, although multistage DJM is formally a one-step integrator.	[90], [16]
4. Consistent comparison	Existing comparisons differ in truncation order, error norm, test problem and cost reporting; a controlled comparison with spectral or stiff-solver reference solutions is required.	[114], [113], [28], [46], [98]
5. Hybridisation	Hybrids of DJM with transforms, Padé approximants and classical discretisations remain underexplored, although those of ADM, HPM and VIM have extended the domain of validity.	[23], [62], [63], [65]

11. Conclusions and future directions

Methods for nonlinear differential equations divide into a mature body of discretisation and integration theory, with rigorous convergence and stability frameworks, and a more recent family of semi-analytical series methods that trade a thinner theory for closed-form, low-cost solutions. The Daftardar-Gejji–Jafari method is the most economical member of the latter family, requiring neither Adomian polynomials, homotopy parameters nor Lagrange multipliers, and it has been applied to integer- and fractional-order ODEs, PDEs and integro-differential equations. Its principal weaknesses, a limited radius of convergence, expression swell and the absence of stability analysis, are shared by the whole family and are most visible for large domains, stiff problems and chaotic or soliton dynamics.

Four directions follow. First, a systematically developed multistage DJM, with step-size control and stability-region analysis, would make the method a one-step integrator with analytically derived coefficients and open stiff and chaotic problems to it. Second, hybridisation with Padé approximants, transforms and classical spatial discretisations deserves the attention already given to ADM, HPM and VIM. Third, convergence theory should move from sufficient conditions towards sharper, verifiable bounds and residual-based error estimates that do not require an exact solution. Fourth, comparative studies should be controlled, using identical problems and accuracy targets, reporting CPU time, expression size and error norms, and referencing high-resolution spectral or stiff-solver solutions. The Fisher, NWS and Boussinesq equations, together with Lorenz-type chaotic systems, provide a suitable test suite for such work.

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