

THE KILBAS–SAIGO–HANKEL TRANSFORM AND ITS L^2 INVERSION THEORY

¹Balasaheb B. Waphare

¹Director and Professor of Mathematics

¹Department of Mathematics

¹MAEER's MIT Arts, Commerce and Science College (MITACSC), Alandi (D), Pune 412105, Maharashtra, India

ORCID: 0000-0003-0976-0615

Email: balasahebwapahare@gmail.com

Abstract : We introduce the integral transform on the half-line whose kernel is the Hankel kernel modulated multiplicatively by the Kilbas–Saigo function, $K_{\nu,\rho,\lambda}^{(\alpha,m,l)}(x,t) = \sqrt{xt} J_\nu(xt) E_{\alpha,m,l}(-\lambda(xt)^{-\rho})$ for $x, t > 0$, where $E_{\alpha,m,l}$ is the three-parameter Kilbas–Saigo function, J_ν the Bessel function of the first kind, $\nu \geq -1/2$, $\rho > 1/2$, $\lambda \geq 0$, and the Kilbas–Saigo parameters lie on the completely monotone line $\alpha \in (0, 1]$, $m > 0$, $l = m - 1$. Under the stated assumptions the envelope suppresses the low- xt region — for $\lambda > 0$ the full kernel tends to zero as $xt \rightarrow 0^+$ — while the kernel agrees with the Hankel kernel to first order at high frequency, with the exact structure constant $c_1 = \Gamma(1 + \alpha l) / \Gamma(1 + \alpha(l + 1))$. Using the Bernstein (complete-monotonicity) representation of the envelope — available by a theorem of Boudabsa, Simon and Vallois — we prove two-sided second-order bounds $1 - c_1 u \leq E_{\alpha,m,l}(-u) \leq 1 - c_1 u + c_2 u^2$, deduce a fully explicit Schur bound $\theta(\lambda) = C_\nu (c_1 \lambda)^{1/(2\rho)} (2 + (\rho - 1/2)^{-1})$ for the L^2 operator norm of the deviation from the Hankel transform, and obtain: boundedness of the transform on $L^2(0, \infty)$; two-sided frame-type inequalities $(1 - \theta) \|f\|_2 \leq \| \mathcal{K}_\lambda f \|_2 \leq (1 + \theta) \|f\|_2$; a quasi-Parseval identity with explicit defect; and an inversion theorem with a Neumann-series inversion formula, valid below the explicit threshold $\lambda^* = [C_\nu (2 + (\rho - 1/2)^{-1})]^{-2\rho} / c_1$. Complete proofs are given for every statement. A worked application — the certified reconstruction of a profile from Hankel data attenuated at low frequency by a Kilbas–Saigo relaxation law — demonstrates the inversion theorem numerically, with measured geometric error decay well inside the certified rate. A comparison table locates the transform relative to the Hankel, fractional Hankel, H -, Saigo, and Mittag-Leffler/Prabhakar-type operators. For the illustrative parameter set $(\alpha, m, l, \nu, \rho) = (3/4, 3/2, 1/2, 1, 1)$ every constant is certified numerically at 30–160 significant digits: $c_1 = 0.839024709118146 \dots$, $c_2 = 0.437519421177344 \dots$, $C_1 \leq 0.8250318$, and $\lambda^* = 0.10944$. To the best of the author's knowledge, after the dated literature search documented in Section VII, the Kilbas–Saigo function has not previously appeared as the kernel, or kernel envelope, of an integral transform; the scope and limits of that search are stated explicitly there. A reference implementation reproducing every constant and experiment is included as Appendix B.

IndexTerms - Kilbas–Saigo function, Hankel transform, integral transform, complete monotonicity, Bernstein representation, Schur test, inversion, Neumann series, numerical certification.

MSC Classification - 44A15, 44A20, 33E12, 33C10, 47G10, 26A48.

I. INTRODUCTION

1.1 Background

The Kilbas–Saigo function is the three-parameter entire function

$$E_{\alpha,m,l}(z) = 1 + \sum_{n \geq 1} c_n z^n, \quad c_n = \prod_{k=1}^n \frac{\Gamma(1 + \alpha((k-1)m + l))}{\Gamma(1 + \alpha((k-1)m + l + 1))},$$

with parameters $\alpha, m > 0$ and $l > -1/\alpha$ (so that every Gamma argument in (1) is positive). It was introduced by Kilbas and Saigo [1] as the solution of an Abel–Volterra integral equation on the half-line, and has since become a standard object in fractional calculus: it solves fractional differential equations with power-type coefficients, models anomalous relaxation [4], and, on the parameter line $l = m - 1$, arises in the theory of fractional extreme distributions [2]. For $m = 1$ the product in (1) telescopes to a two-parameter Mittag-Leffler function (Proposition 5 below); for general m the function is not reducible to the Mittag-Leffler or Prabhakar [7, 8] families. Its analytic theory on the negative half-line was substantially developed by Boudabsa and Simon [3], who characterized complete monotonicity, obtained the exact asymptotics at $-\infty$, and derived uniform hyperbolic bounds; earlier, Boudabsa, Simon and Vallois [2] had proved that

$$x \mapsto E_{\alpha,m,m-1}(-x) \text{ is completely monotone on } [0, \infty) \text{ for every } \alpha \in (0, 1], m > 0,$$

extending Pollard's classical theorem for the one-parameter Mittag-Leffler function.

On the transform side, the Hankel transform

$$(H_\nu f)(x) = \int_0^\infty \sqrt{xt} J_\nu(xt) f(t) dt, \quad \nu \geq -1/2,$$

is, by Hankel's theorem, a self-reciprocal unitary operator on $L^2(0, \infty)$ [9, 10]. A substantial literature perturbs, deforms, or generalizes (3) — fractional Hankel transforms [15, 16], Bessel wavelets, index transforms, kernels of Fourier type in the sense of Titchmarsh's Chapter VIII [10] — and, in a different direction, the fractional-calculus literature attaches the names of Kilbas and Saigo to the Saigo fractional-integral operators, whose kernels are Gauss hypergeometric functions [6, 12]. To the best of the author's knowledge, after the search documented in Section VII (whose scope and limits are stated there), the Kilbas–Saigo function (1) itself does not appear to have been used as the kernel, or as a multiplicative kernel envelope, of an integral transform. The present paper develops exactly such a construction.

1.2 The transform and the main results

For $\nu \geq -1/2$, $\rho > 0$, $\lambda \geq 0$, and Kilbas–Saigo parameters on the completely monotone line (2), define the Kilbas–Saigo–Hankel (KSH) kernel and transform

$$\mathcal{K}_\lambda(x, t) := \sqrt{xt} J_\nu(xt) E_{\alpha, m, l}(-\lambda(xt)^{-\rho}), \quad (\mathcal{K}_\lambda f)(x) := \int_0^\infty \mathcal{K}_\lambda(x, t) f(t) dt.$$

The kernel and the integral in (4) are formally defined for every $\rho > 0$; however, all L^2 boundedness and inversion results of this paper assume the standing hypothesis (K3) of Section II, i.e. $\rho > 1/2$ (the regime $\rho \in (0, 1/2]$ is open problem (ii) of Section VIII). At $\lambda = 0$ the envelope is identically 1 and $\mathcal{K}_0 = H_\nu$. For $\lambda > 0$ the envelope argument $-\lambda(xt)^{-\rho}$ tends to 0^- at high frequency and to $-\infty$ at the origin of the phase variable; by (2) the envelope increases from 0 to 1 along the ray, so the kernel is a Hankel kernel with its low- xt content completely monotonically suppressed and its high- xt content asymptotically untouched (Fig. 2). The design objective is a transform that retains the Hankel transform's L^2 analysis at high frequency while carrying a tunable, special-function-controlled regularization at low frequency — with every constant of the theory exact or certified.

The main results, each proved in full, are: entirety and coefficient decay of the series (1) (Lemma 4); the reduction lattice (Proposition 5); the Bernstein representation with exact moment identities (Lemma 6); the two-sided second-order envelope law (Proposition 7); the kernel bounds (Lemma 8); the Schur test and the explicit deviation bound $\theta(\lambda)$ (Lemmas 9 and 10); L^2 boundedness and two-sided frame inequalities (Theorem 11); the quasi-Parseval identity with explicit defect (Proposition 12); and the inversion theorem with Neumann formula below the explicit threshold λ^* (Theorem 13). Section V certifies every constant at 30–160 digits and validates the operator-level statements by quadrature; Section VI gives a worked application of the inversion theorem, with measured error decay; Section VII contains the novelty discussion with a dated search; Section VIII lists open problems, of which the sharp deviation norm (our Schur bound is deliberately crude) is the most natural.

Nothing in the abstract functional analysis below is new — the Schur test, the Neumann series, and Hankel unitarity are classical, and we include their proofs (except Hankel's theorem, which we cite) only to make the paper self-contained with explicit constants. What is claimed as new, in the qualified sense made precise in Section VII, is the object (4), the identification of its exact structure constants, the use of the Kilbas–Saigo complete-monotonicity theorem (2) as the positivity engine of a transform theory, and the certified quantitative form of every statement.

1.3 Position within the transform literature

Table 1 locates the KSH transform relative to the principal families of half-line transforms and fractional-calculus operators with which it could be confused, and records the structural feature that separates it from each. Two distinctions deserve emphasis. First, in the Saigo fractional-integral operators and in the H-transform, a special function (Gauss hypergeometric, respectively Fox H) is the kernel of a fractional integral or of a Mellin-convolution operator; in the KSH transform the special function is a multiplicative envelope on an oscillatory unitary kernel, and the resulting operator is a perturbation of a unitary, not a fractional integral. Second, in the Prabhakar theory the three-parameter Mittag-Leffler function generates convolution-type fractional operators [7, 8]; the KSH construction is not of convolution type in the additive variable — its kernel depends on the product xt , i.e. it is of Fourier–Hankel (Mellin-convolution) type — and its envelope is the Kilbas–Saigo function, which for $m \neq 1$ lies outside the Mittag-Leffler and Prabhakar families (Proposition 5).

Table 1 The KSH transform against the transform and fractional-operator families nearest to it. Citations are to representative sources, not exhaustive ones.

Operator family	Kernel	L^2 structure	Relation to KSH (4)
Hankel transform [9, 10]	$\sqrt{xt} J_\nu(xt)$	unitary, self-reciprocal	the $\lambda = 0$ member of the KSH family
Fractional Hankel transform [15, 16]	chirp-modulated Bessel kernel (fractional power H_ν^θ)	unitary for each order	deforms the operator power; KSH deforms the kernel amplitude, unitarity replaced by frame bounds (17)
H -transform [12]	Fox H -function $H_{p,q}^{m,n}(xt)$	Mellin-multiplier theory on weighted spaces	kernel is an H -function; the KS envelope, for $m \neq 1$, is not an H -function of its argument, and KSH is a perturbation of a unitary
Saigo fractional integrals [6, 12]	Gauss ${}_2F_1$ kernel, Abel-type factor	bounded fractional integrals, not unitary	fractional integral operators; no Bessel oscillation, no Plancherel-type theory; only the names coincide (Sec. 7)
Mittag-Leffler kernels [5]	$E_{\alpha, \beta}$ in convolution or integral-equation kernels	operator theory of relaxation/convolution type	the $m = 1$ face of the KS envelope telescopes to $E_{\alpha, \beta}$ (Prop. 5); the resulting Mittag-Leffler–Hankel transform also appears to be new
Prabhakar-type operators [7, 8]	$E_{\alpha, \beta}^\gamma$ in Abel/convolution kernels	fractional calculus, semigroup properties	convolution type in the additive variable; KSH is of product (xt) type, and for $m \neq 1$ the KS function is not a Prabhakar function
KSH transform (this paper)	$\sqrt{xt} J_\nu(xt) \times E_{\alpha, m, l}(-\lambda(xt)^{-\rho})$	bounded; two-sided frame bounds; Neumann-invertible for $\lambda < \lambda^*$	—

II. PRELIMINARIES AND STANDING HYPOTHESES

Definition 1 (Kilbas–Saigo function). $E_{\alpha, m, l}$ is defined by (1) for $\alpha, m > 0$, $l > -1/\alpha$; in particular $c_0 = 1$ and $E_{\alpha, m, l}(0) = 1$. Entirety is proved in Lemma 4 below.

Definition 2 (Structure constants). With c_n as in (1),

$$c_1 = \frac{\Gamma(1 + \alpha l)}{\Gamma(1 + \alpha(l + 1))}, \quad c_2 = c_1 \cdot \frac{\Gamma(1 + \alpha(m + l))}{\Gamma(1 + \alpha(m + l + 1))}.$$

Throughout the paper we assume:

(K1) $\alpha \in (0, 1]$, $m > 0$, $l = m - 1$ (the completely monotone line; see Remark 3);

- (K2) $\nu \geq -1/2$;
(K3) $\rho > 1/2$ and $\lambda \geq 0$.

Remark 3. (K1) is used only through the complete monotonicity (2). Boudabsa and Simon [3] characterize the full parameter region in which $E_{\alpha,m,l}(-\cdot)$ is completely monotone; every result of this paper extends verbatim to that region, with the same proofs. We fix the line $l = m - 1$ because there the complete monotonicity is guaranteed by the citable closed statement (2) for all $\alpha \in (0,1]$, $m > 0$, which keeps the hypotheses of our theorems checkable at a glance. Note that on this line the condition $l > -1/\alpha$ holds automatically, since $l = m - 1 > -1 \geq -1/\alpha$.

We shall use two classical facts. First, Hankel's theorem [9, Ch. XIV], [10, Ch. VIII]: for $\nu \geq -1/2$ the operator H_ν of (3), initially defined by the absolutely convergent integral on $L^1 \cap L^2$, extends to a self-reciprocal ($H_\nu^2 = I$) unitary operator on $L^2(0, \infty)$. Second, Bernstein's theorem [11, Thm. 1.4]: a function φ is completely monotone on $(0, \infty)$ if and only if it is the Laplace transform of a nonnegative Borel measure μ on $[0, \infty)$, $\varphi(u) = \int_0^\infty e^{-us} \mu(ds)$; the measure is then uniquely determined.

2.1 Entirety and the reduction lattice

Lemma 4 (Entirety and coefficient decay). Let $\alpha, m > 0$, $l > -1/\alpha$. Then $c_n > 0$ for all $n \geq 0$, and

$$\frac{c_{n+1}}{c_n} = \frac{\Gamma(1 + \alpha(nm + l))}{\Gamma(1 + \alpha(nm + l + 1))} \leq 2(1 + \alpha(nm + l))^{-\alpha} \rightarrow 0 \quad (n \rightarrow \infty),$$

the inequality holding for $\alpha \in (0,1]$ whenever $1 + \alpha(nm + l) \geq 1$, that is, whenever $nm + l \geq 0$ — hence for all $n \geq n_0 := \max\{0, \lceil -l/m \rceil\}$, and in particular for all $n \geq 0$ when $l \geq 0$ (on the line $l = m - 1$ of (K1), for all $n \geq 0$ when $m \geq 1$, and for all $n \geq \lceil (1 - m)/m \rceil$ when $0 < m < 1$). The eventual decay of the ratio is all that is required: consequently the series (1) converges absolutely for every $z \in \mathbb{C}$, locally uniformly, and $E_{\alpha,m,l}$ is entire.

Proof. Positivity. Every Gamma argument $1 + \alpha((k - 1)m + l)$ and $1 + \alpha((k - 1)m + l + 1)$ appearing in (1) is positive: the smallest of them is $1 + \alpha l > 0$ by the hypothesis $l > -1/\alpha$, and the arguments increase in k since $m > 0$. Each factor of c_n is therefore a ratio of values of Γ at positive arguments, hence positive; so $c_n > 0$.

The ratio identity. Passing from c_n to c_{n+1} appends to the product exactly the factor with index $k = n + 1$, whose numerator argument is $1 + \alpha(nm + l)$ and denominator argument is $1 + \alpha(nm + l + 1)$; this is the equality in (6).

The quantitative bound. Write $x := 1 + \alpha(nm + l)$ and $a := \alpha$; we give the argument for $a \in (0,1]$, the case used in this paper (for $a > 1$ split off the integer part and iterate). Assume $x \geq 1$, which holds precisely when $nm + l \geq 0$, hence for every $n \geq n_0$; the finitely many indices $n < n_0$ are irrelevant to the decay statement and to entirety. The Gamma function is logarithmically convex on $(0, \infty)$ (Bohr–Mollerup; equivalently, Hölder's inequality applied to the Euler integral): for $p > 0$ and $\theta \in [0,1]$,

$$\Gamma(p + \theta) = \int_0^\infty (t^{p-1}e^{-t})^{1-\theta} (t^p e^{-t})^\theta dt \leq \Gamma(p)^{1-\theta} \Gamma(p + 1)^\theta.$$

Apply this with $p = x + a$ and $\theta = 1 - a$, so that $p + \theta = x + 1$:

$$x \Gamma(x) = \Gamma(x + 1) \leq \Gamma(x + a)^a \Gamma(x + a + 1)^{1-a} = \Gamma(x + a) (x + a)^{1-a},$$

using $\Gamma(x + a + 1) = (x + a)\Gamma(x + a)$. Rearranging, and using $a \leq 1 \leq x$ so that $x + a \leq 2x$,

$$\frac{\Gamma(x)}{\Gamma(x + a)} \leq \frac{(x + a)^{1-a}}{x} \leq 2^{1-a} x^{-a} \leq 2 x^{-a}.$$

Since $x = 1 + \alpha(nm + l) \rightarrow \infty$, the ratio $c_{n+1}/c_n \leq 2x^{-a} \rightarrow 0$ along $n \geq n_0$. (Wendel's classical two-sided inequality $(x/(x + a))^{1-a} \leq \Gamma(x + a)/(x^a \Gamma(x)) \leq 1$ for $0 \leq a \leq 1$, $x > 0$ [14], yields $\Gamma(x)/\Gamma(x + a) \leq x^{-a}(1 + a/x)^{1-a}$ for every $x > 0$, hence the decay of the ratio for all $n \geq 0$ without the restriction $x \geq 1$; only the eventual decay is used below, for which the elementary bound just proved suffices.)

Entirety. Fix $z \in \mathbb{C}$, $z \neq 0$. By (6), for $n \geq n_0$ one has $|c_{n+1}z^{n+1}|/|c_n z^n| = |z| c_{n+1}/c_n \rightarrow 0 < 1$, so the series converges absolutely by the ratio test (the finitely many initial terms do not affect convergence). On any disc $|z| \leq R$ the terms are dominated by $c_n R^n$, a convergent series, so convergence is uniform there by the Weierstrass M -test, and the sum is analytic on every disc, i.e. entire [1, 3]. ◻

Proposition 5 (Reduction lattice). (i) If $\lambda = 0$ then $\mathcal{K}_\lambda = H_\nu$, the Hankel transform. (ii) If $m = 1$, the product in (1) telescopes and

$$E_{\alpha,1,l}(z) = \Gamma(1 + \alpha l) E_{\alpha,1+\alpha l}(z), \quad E_{\alpha,\beta}(z) := \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

the two-parameter Mittag-Leffler function [5] up to the constant $\Gamma(1 + \alpha l)$; the KSH family thus contains a Mittag-Leffler–Hankel transform as a degenerate face. (iii) If $\alpha = m = 1$, $l = 0$, then $E_{1,1,0}(z) = e^z$, the kernel envelope is $e^{-\lambda(xt)^{-\rho}}$, and $c_1 = 1$, $c_2 = 1/2$.

Proof. (i) At $\lambda = 0$ the envelope argument is 0 and $E_{\alpha,m,l}(0) = 1$ by the normalization $c_0 = 1$; the kernel (4) reduces to the Hankel kernel and the transforms coincide on their common domain of definition, hence as L^2 operators.

(ii) For $m = 1$ the k -th factor of c_n is $\Gamma(1 + \alpha(k - 1 + l))/\Gamma(1 + \alpha(k + l))$. The denominator of the k -th factor equals the numerator of the $(k + 1)$ -st factor, so the product telescopes:

$$c_n = \frac{\Gamma(1 + \alpha l)}{\Gamma(1 + \alpha(1 + l))} \cdot \frac{\Gamma(1 + \alpha(1 + l))}{\Gamma(1 + \alpha(2 + l))} \cdots \frac{\Gamma(1 + \alpha(n - 1 + l))}{\Gamma(1 + \alpha(n + l))} = \frac{\Gamma(1 + \alpha l)}{\Gamma(\alpha n + 1 + \alpha l)}.$$

Substituting into (1) (including the $n = 0$ term, for which the empty product is $1 = \Gamma(1 + \alpha l)/\Gamma(1 + \alpha l)$) gives $E_{\alpha,1,l}(z) = \Gamma(1 + \alpha l) \sum_{n \geq 0} z^n / \Gamma(\alpha n + 1 + \alpha l)$, which is (7) with $\beta = 1 + \alpha l$.

(iii) With $\alpha = m = 1, l = 0$, the telescoped coefficient is $c_n = \Gamma(1)/\Gamma(1 + n) = 1/n!$, so $E_{1,1,0}(z) = e^z$ and the envelope is $e^{-\lambda(xt)^{-\rho}}$. From (5), $c_1 = \Gamma(1)/\Gamma(2) = 1$ and $c_2 = c_1 \cdot \Gamma(2)/\Gamma(3) = 1/2$, matching the Taylor coefficients of e^z , as they must. ▀

III. THE KERNEL: STRUCTURAL BOUNDS AND THE TWO-TERM LAW

Lemma 6 (Bernstein representation and moment identities). Under (K1) there is a unique probability measure μ on $[0, \infty)$ with

$$E_{\alpha,m,l}(-u) = \int_0^\infty e^{-us} \mu(ds), \quad u \geq 0,$$

and its first two moments are finite and given exactly by the structure constants:

$$\int_0^\infty s \mu(ds) = c_1, \quad \int_0^\infty s^2 \mu(ds) = 2c_2.$$

Moreover $\mu(\{0\}) = 0$; equivalently $E_{\alpha,m,l}(-u) \rightarrow 0$ as $u \rightarrow \infty$.

Proof. Step 1: existence, uniqueness, and total mass. By (2) the function $u \mapsto E_{\alpha,m,l}(-u)$ is completely monotone; by Bernstein's theorem [11, Thm. 1.4] there is a unique nonnegative Borel measure μ on $[0, \infty)$ with (8) for $u > 0$. Let $u \downarrow 0$: the integrands e^{-us} increase pointwise to 1 as u decreases, so by the monotone convergence theorem $\lim_{u \downarrow 0} E_{\alpha,m,l}(-u) = \mu([0, \infty))$. The left-hand limit equals $E_{\alpha,m,l}(0) = 1$ because $E_{\alpha,m,l}$ is entire (Lemma 4), hence continuous at the origin. Thus μ is a probability measure, and (8) extends to $u = 0$.

Step 2: the first moment. For $u > 0$, using (8) and $\mu([0, \infty)) = 1$,

$$\frac{1 - E_{\alpha,m,l}(-u)}{u} = \int_0^\infty \frac{1 - e^{-us}}{u} \mu(ds).$$

The function $h(v) := (1 - e^{-v})/v$ (with $h(0) := 1$) is positive and decreasing on $[0, \infty)$: its derivative has the sign of $e^{-v}(1 + v) - 1$, which is ≤ 0 because $e^v \geq 1 + v$. Writing the integrand of (10) as $s h(us)$, we see that for each fixed $s > 0$ it increases as $u \downarrow 0$ (since h is decreasing and $us \downarrow 0$), with pointwise limit $s h(0) = s$; for $s = 0$ it vanishes identically. By the monotone convergence theorem applied along any sequence $u_k \downarrow 0$,

$$\lim_{u \downarrow 0} \frac{1 - E_{\alpha,m,l}(-u)}{u} = \int_0^\infty s \mu(ds) \in [0, \infty].$$

Independently, Lemma 4 lets us compute the same limit from the entire series (1): $E_{\alpha,m,l}(-u) = 1 - c_1 u + c_2 u^2 - \dots$, so $(1 - E_{\alpha,m,l}(-u))/u = c_1 - c_2 u + \dots \rightarrow c_1$, a finite limit. Equating the two evaluations gives $\int s \mu(ds) = c_1 < \infty$.

Step 3: the second moment. We use the order-two Taylor formula with integral remainder for the exponential: for every $x \geq 0$,

$$e^{-x} = 1 - x + x^2 \int_0^1 (1 - \theta) e^{-\theta x} d\theta,$$

obtained by two integrations by parts of $\int_0^x (x - t) e^{-t} dt$, or directly by verifying that both sides agree at $x = 0$ together with their first derivatives and have equal second derivatives. Consequently, for $u > 0$,

$$\frac{E_{\alpha,m,l}(-u) - 1 + c_1 u}{u^2} = \int_0^\infty \frac{e^{-us} - 1 + us}{u^2} \mu(ds) = \int_0^\infty s^2 \left(\int_0^1 (1 - \theta) e^{-\theta us} d\theta \right) \mu(ds),$$

where the first equality combines (8), $\mu([0, \infty)) = 1$, and Step 2, and the second is (11) at $x = us$. For each fixed $s \geq 0$ the inner factor $\int_0^1 (1 - \theta) e^{-\theta us} d\theta$ is nonnegative and increases as $u \downarrow 0$ (the integrand increases pointwise in θ as u decreases), with limit $\int_0^1 (1 - \theta) d\theta = 1/2$. By monotone convergence once more,

$$\lim_{u \downarrow 0} \frac{E_{\alpha,m,l}(-u) - 1 + c_1 u}{u^2} = \frac{1}{2} \int_0^\infty s^2 \mu(ds) \in [0, \infty],$$

while the series gives the finite value c_2 for the same limit. Hence $\int s^2 \mu(ds) = 2c_2 < \infty$.

Step 4: no atom at the origin. For every $s \geq 0$, $e^{-us} \rightarrow \mathbf{1}_{\{0\}}(s)$ as $u \rightarrow \infty$, with $0 \leq e^{-us} \leq 1$ and μ finite; dominated convergence gives $\lim_{u \rightarrow \infty} E_{\alpha,m,l}(-u) = \mu(\{0\})$. By the exact asymptotics of the Kilbas–Saigo function at $-\infty$ established in [3], $E_{\alpha,m,l}(-u) \rightarrow 0$; equivalently, in the probabilistic representation underlying (2), μ is the law of an almost surely strictly positive integral functional of the α -stable subordinator [2, 3], whence $\mu(\{0\}) = 0$ directly. Either argument closes the proof. ▀

Proposition 7 (Two-sided second-order law and monotonicity). Under (K1), for every $u \geq 0$:

$$1 - c_1 u \leq E_{\alpha,m,l}(-u) \leq 1 - c_1 u + c_2 u^2, \\ 1 - E_{\alpha,m,l}(-u) \leq \min\{1, c_1 u\},$$

and $E_{\alpha,m,l}(-u)$ is strictly positive and strictly decreasing on $[0, \infty)$, with $E_{\alpha,m,l}(0) = 1$ and $E_{\alpha,m,l}(-u) \rightarrow 0$ as $u \rightarrow \infty$.

Proof. Proof of (12). By (8), (9), and the exact remainder identity (11) at $x = us$,

$$E_{\alpha,m,l}(-u) - (1 - c_1 u) = \int_0^\infty (e^{-us} - 1 + us) \mu(ds) = u^2 \int_0^\infty s^2 \left(\int_0^1 (1 - \theta) e^{-\theta us} d\theta \right) \mu(ds).$$

The inner factor obeys the two-sided elementary bound $0 \leq \int_0^1 (1 - \theta) e^{-\theta us} d\theta \leq \int_0^1 (1 - \theta) d\theta = 1/2$, the upper estimate using $e^{-\theta us} \leq 1$. Hence $0 \leq E_{\alpha, m, l}(-u) - (1 - c_1 u) \leq 1/2 u^2 \int_0^\infty s^2 \mu(ds) = c_2 u^2$ by (9), which is precisely (12).

Proof of (13). From (8) and $\mu([0, \infty)) = 1$, $1 - E_{\alpha, m, l}(-u) = \int_0^\infty (1 - e^{-us}) \mu(ds)$. For $v \geq 0$ both elementary bounds $1 - e^{-v} \leq 1$ and $1 - e^{-v} \leq v$ hold (the second is the convexity inequality $e^{-v} \geq 1 - v$). Integrating the first gives $1 - E_{\alpha, m, l}(-u) \leq 1$; integrating the second gives $1 - E_{\alpha, m, l}(-u) \leq u \int s \mu(ds) = c_1 u$ by (9). Taking the minimum yields (13).

Positivity, strict decrease, limits. Positivity is immediate from (8): the integrand is strictly positive and μ is a probability measure. For strict decrease, we differentiate (8) under the integral sign on $(0, \infty)$. Justification: fix $u_0 > 0$ and $u \geq u_0$; the difference quotients $(e^{-(u+h)s} - e^{-us})/h$ are, by the mean value theorem, of modulus $\leq s e^{-(u_0/2)s}$ for $|h| \leq u_0/2$, and $s e^{-(u_0/2)s}$ is bounded on $[0, \infty)$, hence μ -integrable (μ finite); dominated convergence applies and

$$-\frac{d}{du} E_{\alpha, m, l}(-u) = \int_0^\infty s e^{-us} \mu(ds) > 0 \quad (u > 0),$$

strictly, because μ charges $(0, \infty)$: indeed $\int s \mu(ds) = c_1 > 0$ by (9) and positivity of the Gamma ratio (5), which forces $\mu((0, \infty)) > 0$. Strict decrease on $[0, \infty)$ follows by continuity at 0. The boundary values are $E_{\alpha, m, l}(0) = 1$ (normalization) and $\lim_{u \rightarrow \infty} E_{\alpha, m, l}(-u) = \mu(\{0\}) = 0$ (Step 4 of Lemma 6). *

Lemma 8 (Kernel bounds). Let $C_v := \sup_{s>0} \sqrt{s} |J_v(s)|$. Then under (K1)–(K3):

(i) $C_v < \infty$, and $|\mathcal{K}_\lambda(x, t)| \leq C_v$ for all $x, t > 0$; consequently $\mathcal{K}_\lambda f$ is defined by an absolutely convergent integral for every $f \in L^1(0, \infty)$, with $\|\mathcal{K}_\lambda f\|_\infty \leq C_v \|f\|_1$;

(ii) the deviation kernel $\mathcal{R}_\lambda(x, t) := \mathcal{K}_\lambda(x, t) - \sqrt{xt} J_v(xt)$ satisfies

$$|\mathcal{R}_\lambda(x, t)| \leq C_v \min\{1, c_1 \lambda(xt)^{-\rho}\} =: g(xt),$$

and for every $f \in L^2(0, \infty)$ the integral $\int_0^\infty \mathcal{R}_\lambda(x, t) f(t) dt$ converges absolutely for every $x > 0$;

(iii) the kernel obeys the two-term high-frequency law with second-order control:

$$\mathcal{K}_\lambda(x, t) = \sqrt{xt} J_v(xt) (1 - c_1 \lambda(xt)^{-\rho} + \theta(x, t) c_2 \lambda^2(xt)^{-2\rho}), \quad 0 \leq \theta(x, t) \leq 1.$$

Proof. (i) **Finiteness of C_v .** The function $\varphi(s) := \sqrt{s} |J_v(s)|$ is continuous on $(0, \infty)$; we bound it near each end of the half-line.

Near $s = 0$. For $0 < s \leq 1$ consider the Bessel series $J_v(s) = \sum_{k \geq 0} (-1)^k (s/2)^{v+2k} / (k! \Gamma(v+k+1))$. The ratio of the moduli of consecutive terms is

$$\frac{(s/2)^2}{(k+1)(v+k+1)} \leq \frac{1/4}{1 \cdot (v+1)} \leq \frac{1/4}{1/2} = \frac{1}{2} < 1 \quad (v \geq -1/2, k \geq 0, 0 < s \leq 1),$$

so the series is alternating with strictly decreasing term moduli, and the alternating series estimate bounds the sum by its first term: $|J_v(s)| \leq (s/2)^v / \Gamma(v+1)$. Hence $\varphi(s) \leq s^{v+1/2} / (2^v \Gamma(v+1)) \leq 2^{-v} / \Gamma(v+1)$ for $0 < s \leq 1$, using $v+1/2 \geq 0$ so that $s^{v+1/2} \leq 1$ on $(0, 1]$.

Near $s = \infty$. By the classical asymptotic expansion of the Bessel function [9, §7.21], for $s \geq 1$, $\sqrt{s} J_v(s) = \sqrt{2/\pi} \cos(s - \pi\nu/2 - \pi/4) + O(s^{-1})$, with an O -constant depending only on ν ; hence φ is bounded on $[1, \infty)$.

A continuous function bounded on $(0, 1]$ and on $[1, \infty)$ is bounded on $(0, \infty)$; thus $C_v < \infty$. The kernel bound now follows from Proposition 7: $0 < E_{\alpha, m, l}(-\lambda(xt)^{-\rho}) \leq 1$, so $|\mathcal{K}_\lambda(x, t)| \leq \sqrt{xt} |J_v(xt)| \leq C_v$. For $f \in L^1$, $\int_0^\infty |\mathcal{K}_\lambda(x, t) f(t)| dt \leq C_v \|f\|_1 < \infty$, uniformly in x , giving absolute convergence and the stated sup bound.

(ii) By the definition of $\mathcal{R}_\lambda$ and the bound (13) evaluated at $u = \lambda(xt)^{-\rho}$,

$$|\mathcal{R}_\lambda(x, t)| = \sqrt{xt} |J_v(xt)| \cdot |1 - E_{\alpha, m, l}(-\lambda(xt)^{-\rho})| \leq C_v \min\{1, c_1 \lambda(xt)^{-\rho}\},$$

which is (14). For absolute convergence against $f \in L^2(0, \infty)$, Cauchy–Schwarz gives $\int_0^\infty |\mathcal{R}_\lambda(x, t) f(t)| dt \leq (\int_0^\infty g(xt)^2 dt)^{1/2} \|f\|_2$, and by the substitution $u = xt$ (so $dt = du/x$), with $u_0 := (c_1 \lambda)^{1/\rho}$,

$$\int_0^\infty g(xt)^2 dt = \frac{1}{x} \int_0^\infty g(u)^2 du = \frac{C_v^2}{x} \left(\int_0^{u_0} 1 du + (c_1 \lambda)^2 \int_{u_0}^\infty u^{-2\rho} du \right) = \frac{C_v^2}{x} \left(u_0 + \frac{(c_1 \lambda)^2 u_0^{1-2\rho}}{2\rho - 1} \right) < \infty,$$

the last integral converging because $2\rho > 1$ by (K3).

(iii) Proposition 7 gives, for each $u > 0$, $E_{\alpha, m, l}(-u) = 1 - c_1 u + \theta(u) c_2 u^2$ where $\theta(u) := (E_{\alpha, m, l}(-u) - 1 + c_1 u) / (c_2 u^2) \in [0, 1]$ by the two-sided bound (12) (and $\theta(0) := 0$ by continuity). Substituting $u = \lambda(xt)^{-\rho}$ and multiplying by $\sqrt{xt} J_v(xt)$ gives the display, with $\theta(x, t) := \theta(\lambda(xt)^{-\rho})$. *

IV. L^2 THEORY: BOUNDEDNESS, FRAME INEQUALITIES, QUASI-PARSEVAL, INVERSION

We first record the Schur test in the precise form we use, with proof, so that the constant tracked through it is explicit.

Lemma 9 (Schur test). Let $k: (0, \infty)^2 \rightarrow \mathbb{C}$ be measurable and $w: (0, \infty) \rightarrow (0, \infty)$ measurable. Suppose there are constants $A, B \geq 0$ with

$$\int_0^\infty |k(x, t)| w(t) dt \leq A w(x) \text{ for a.e. } x, \quad \int_0^\infty |k(x, t)| w(x) dx \leq B w(t) \text{ for a.e. } t.$$

Then the integral operator $(Tf)(x) = \int_0^\infty k(x, t) f(t) dt$ is bounded on $L^2(0, \infty)$ with $\|T\|_{L^2 \rightarrow L^2} \leq \sqrt{AB}$, and the defining integral converges absolutely for a.e. x whenever $f \in L^2(0, \infty)$.

Proof. For $f \in L^2(0, \infty)$ and a.e. x , split $|k| = (|k|w(t))^{1/2} \cdot (|k|/w(t))^{1/2}$ and apply the Cauchy–Schwarz inequality in t :

$$\left(\int_0^\infty |k(x, t)| |f(t)| dt \right)^2 \leq \left(\int_0^\infty |k(x, t)| w(t) dt \right) \left(\int_0^\infty \frac{|k(x, t)|}{w(t)} |f(t)|^2 dt \right) \leq A w(x) \int_0^\infty \frac{|k(x, t)|}{w(t)} |f(t)|^2 dt.$$

Integrating in x and using Tonelli's theorem (all integrands nonnegative) and the second row hypothesis,

$$\int_0^\infty \left(\int_0^\infty |k(x, t)| |f(t)| dt \right)^2 dx \leq A \int_0^\infty \frac{|f(t)|^2}{w(t)} \left(\int_0^\infty |k(x, t)| w(x) dx \right) dt \leq AB \|f\|_2^2.$$

The finiteness of the left side forces the inner integral to be finite for a.e. x (absolute convergence), and $\|Tf\|_2^2 \leq AB \|f\|_2^2$ by $|Tf| \leq \int |k||f|$. *

Lemma 10 (Explicit Schur bound for the deviation). Under (K1)–(K3) with $\lambda > 0$, the integral operator $\mathcal{R}_\lambda$ with kernel $\mathcal{R}_\lambda(x, t)$ of Lemma 8 is bounded on $L^2(0, \infty)$, with

$$\|\mathcal{R}_\lambda\|_{L^2 \rightarrow L^2} \leq \theta(\lambda) := C_v (c_1 \lambda)^{\frac{1}{2\rho}} \left(2 + \frac{1}{\rho - \frac{1}{2}} \right).$$

Proof. Apply Lemma 9 with the weight $w(t) = t^{-1/2}$ and the majorant $|\mathcal{R}_\lambda(x, t)| \leq g(xt)$ of (14). Because the majorant depends on the product xt only, both row conditions reduce to the same one-dimensional integral: substituting $u = xt$ ($dt = du/x$),

$$\int_0^\infty g(xt) t^{-1/2} dt = \int_0^\infty g(u) \left(\frac{u}{x} \right)^{-1/2} \frac{du}{x} = x^{-1/2} \int_0^\infty g(u) u^{-1/2} du =: x^{-1/2} I,$$

so the first row condition holds with $A = I$; by the symmetry $g(xt) = g(tx)$, the second holds with $B = I$; hence $\|\mathcal{R}_\lambda\| \leq I$. It remains to compute I . With $u_0 := (c_1 \lambda)^{1/\rho}$ (the crossover point where $c_1 \lambda u^{-\rho} = 1$),

$$I = C_v \left(\int_0^{u_0} u^{-1/2} du + c_1 \lambda \int_{u_0}^\infty u^{-\rho-\frac{1}{2}} du \right) = C_v \left(2u_0^{1/2} + \frac{c_1 \lambda u_0^{\frac{1}{2}-\rho}}{\rho - \frac{1}{2}} \right),$$

the second integral converging precisely because $\rho > 1/2$ (K3). Finally, $u_0^{1/2} = (c_1 \lambda)^{\frac{1}{2\rho}}$ and $c_1 \lambda u_0^{\frac{1}{2}-\rho} = (c_1 \lambda) \cdot (c_1 \lambda)^{\frac{1/2-\rho}{\rho}} = (c_1 \lambda)^{\frac{\rho+1/2-\rho}{\rho}} = (c_1 \lambda)^{\frac{1}{2\rho}}$, so $I = C_v (c_1 \lambda)^{1/(2\rho)} (2 + (\rho - 1/2)^{-1}) = \theta(\lambda)$, which is (15). *

Theorem 11 (Boundedness and frame inequalities). Under (K1)–(K3), the operator $\mathcal{K}_\lambda$, defined by the absolutely convergent integral (4) on $L^1 \cap L^2$, extends uniquely to a bounded operator on $L^2(0, \infty)$ satisfying

$$\|\mathcal{K}_\lambda\|_{L^2 \rightarrow L^2} \leq 1 + \theta(\lambda),$$

and this extension coincides with $H_v + \mathcal{R}_\lambda$, where H_v is the unitary Hankel transform and $\mathcal{R}_\lambda$ the (everywhere absolutely convergent) deviation operator of Lemma 10. Moreover, for every $f \in L^2(0, \infty)$,

$$(1 - \theta(\lambda)) \|f\|_2 \leq \|\mathcal{K}_\lambda f\|_2 \leq (1 + \theta(\lambda)) \|f\|_2,$$

the lower bound being informative when $\theta(\lambda) < 1$.

Proof. Step 1: decomposition on $L^1 \cap L^2$. For $f \in L^1 \cap L^2$ all three integrals converge absolutely for every $x > 0$: the $\mathcal{K}_\lambda$ - and H_v -integrals by Lemma 8(i) (kernels bounded by C_v , $f \in L^1$), the $\mathcal{R}_\lambda$ -integral by Lemma 8(ii) (even for all $f \in L^2(0, \infty)$). Since the kernels satisfy $\mathcal{K}_\lambda = \sqrt{xt} J_v(xt) + \mathcal{R}_\lambda$ pointwise, we have the pointwise identity $\mathcal{K}_\lambda f = H_v f + \mathcal{R}_\lambda f$ on $L^1 \cap L^2$.

Step 2: boundedness on the dense subspace and extension. By Hankel's theorem, $\|H_v f\|_2 = \|f\|_2$ for $f \in L^1 \cap L^2$ (the integral operator agrees with the unitary extension there [10]); by Lemma 10, $\|\mathcal{R}_\lambda f\|_2 \leq \theta(\lambda) \|f\|_2$. The triangle inequality gives $\|\mathcal{K}_\lambda f\|_2 \leq \|H_v f\|_2 + \|\mathcal{R}_\lambda f\|_2 \leq (1 + \theta(\lambda)) \|f\|_2$ for $f \in L^1 \cap L^2$. Since $L^1 \cap L^2$ is dense in $L^2(0, \infty)$ (it contains all compactly supported square-integrable functions), $\mathcal{K}_\lambda$ extends uniquely to a bounded operator on $L^2(0, \infty)$ with the same norm bound, which is (16); and by continuity of both sides, the identity $\mathcal{K}_\lambda = H_v + \mathcal{R}_\lambda$ persists for the extensions. (For $\mathcal{R}_\lambda$ the extension is redundant: its defining integral already converges absolutely for every $f \in L^2(0, \infty)$ by Lemma 8(ii), and Lemma 9 shows the resulting operator is the bounded one.)

Step 3: the frame inequalities. For $f \in L^2(0, \infty)$, unitarity of H_v and the reverse triangle inequality give $\|\mathcal{K}_\lambda f\|_2 \geq \|H_v f\|_2 - \|\mathcal{R}_\lambda f\|_2 \geq \|f\|_2 - \theta(\lambda) \|f\|_2 = (1 - \theta(\lambda)) \|f\|_2$, which together with Step 2 is (17). *

Proposition 12 (Quasi-Parseval identity with explicit defect). Under (K1)–(K3), for all $f, h \in L^2(0, \infty)$,

$$|\langle \mathcal{K}_\lambda f, \mathcal{K}_\lambda h \rangle - \langle f, h \rangle| \leq (2\theta(\lambda) + \theta(\lambda)^2) \|f\|_2 \|h\|_2,$$

which degenerates to Parseval's identity for the Hankel transform at $\lambda = 0$.

Proof. Using $\mathcal{K}_\lambda = H_v + \mathcal{R}_\lambda$ (Theorem 11) and expanding the inner product,

$$\langle \mathcal{K}_\lambda f, \mathcal{K}_\lambda h \rangle = \langle H_v f, H_v h \rangle + \langle H_v f, \mathcal{R}_\lambda h \rangle + \langle \mathcal{R}_\lambda f, H_v h \rangle + \langle \mathcal{R}_\lambda f, \mathcal{R}_\lambda h \rangle.$$

The first term equals $\langle f, h \rangle$ by unitarity of H_v (polarization of the isometry). Each remaining term is bounded by Cauchy–Schwarz together with $\|H_v\| = 1$ and $\|\mathcal{R}_\lambda\| \leq \theta(\lambda)$: $|\langle H_v f, \mathcal{R}_\lambda h \rangle| \leq \theta \|f\|_2 \|h\|_2$, $|\langle \mathcal{R}_\lambda f, H_v h \rangle| \leq \theta \|f\|_2 \|h\|_2$, and $|\langle \mathcal{R}_\lambda f, \mathcal{R}_\lambda h \rangle| \leq \theta^2 \|f\|_2 \|h\|_2$. Summing the three bounds gives (18). *

Theorem 13 (Inversion below the explicit threshold). Under (K1)–(K3), suppose

$$\lambda < \lambda^* := \frac{1}{c_1} \left[C_v \left(2 + \frac{1}{\rho - \frac{1}{2}} \right) \right]^{-2\rho},$$

equivalently $\theta(\lambda) < 1$. Then $\mathcal{K}_\lambda$ is boundedly invertible on $L^2(0, \infty)$, with $\|\mathcal{K}_\lambda^{-1}\|_{L^2 \rightarrow L^2} \leq (1 - \theta(\lambda))^{-1}$, and the inverse is given by the norm-convergent Neumann series

$$f = \mathcal{K}_\lambda^{-1}g = \sum_{k=0}^{\infty} (-1)^k (H_\nu \mathcal{R}_\lambda)^k H_\nu g, \quad g = \mathcal{K}_\lambda f,$$

whose N -term truncation $f_N := \sum_{k=0}^N (-1)^k (H_\nu \mathcal{R}_\lambda)^k H_\nu g$ satisfies

$$\|f - f_N\|_2 \leq \frac{\theta(\lambda)^{N+1}}{1 - \theta(\lambda)} \|g\|_2.$$

In particular, for every $g \in L^2(0, \infty)$ the equation $\mathcal{K}_\lambda f = g$ has exactly one solution $f \in L^2(0, \infty)$.

Proof. Step 1: factorization. Since H_ν is self-reciprocal, $H_\nu^2 = I$, we may write $\mathcal{K}_\lambda = H_\nu + \mathcal{R}_\lambda = H_\nu(I + H_\nu \mathcal{R}_\lambda)$, because $H_\nu \cdot H_\nu \mathcal{R}_\lambda = H_\nu^2 \mathcal{R}_\lambda = \mathcal{R}_\lambda$. Set $S := H_\nu \mathcal{R}_\lambda$; then $\|S\| \leq \|H_\nu\| \|\mathcal{R}_\lambda\| \leq \theta(\lambda) < 1$ by Lemma 10, unitarity, and the hypothesis (19) — which is exactly the equation $\theta(\lambda) = 1$ of (15) solved for λ , so $\lambda < \lambda^* \Leftrightarrow \theta(\lambda) < 1$ by strict monotonicity of $\lambda \mapsto \theta(\lambda)$.

Step 2: Neumann series for $(I + S)^{-1}$. The partial sums $P_N := \sum_{k=0}^N (-S)^k$ form a Cauchy sequence in operator norm, since for $M > N$, $\|P_M - P_N\| \leq \sum_{k=N+1}^M \|S\|^k \leq \theta^{N+1}/(1 - \theta) \rightarrow 0$; by completeness of the space of bounded operators, $P_N \rightarrow P := \sum_{k=0}^{\infty} (-S)^k$ in norm, with $\|P\| \leq \sum_{k=0}^{\infty} \theta^k = (1 - \theta)^{-1}$. A telescoping computation gives $(I + S)P_N = I - (-S)^{N+1} = P_N(I + S)$, and letting $N \rightarrow \infty$ (using $\|(-S)^{N+1}\| \leq \theta^{N+1} \rightarrow 0$) yields $(I + S)P = P(I + S) = I$: thus $I + S$ is invertible with $(I + S)^{-1} = P$.

Step 3: inversion of $\mathcal{K}_\lambda$ and the formula. From Step 1, $\mathcal{K}_\lambda^{-1} = (I + S)^{-1}H_\nu^{-1} = (I + S)^{-1}H_\nu$ (self-reciprocity again), which is bounded with $\|\mathcal{K}_\lambda^{-1}\| \leq \|(I + S)^{-1}\| \|H_\nu\| \leq (1 - \theta)^{-1}$. Writing out $(I + S)^{-1} = \sum_{k=0}^{\infty} (-1)^k S^k$ gives (20). Existence and uniqueness of the solution of $\mathcal{K}_\lambda f = g$ are equivalent to bounded invertibility.

Step 4: the truncation bound. $f - f_N = \sum_{k>N} (-1)^k S^k H_\nu g$, so $\|f - f_N\|_2 \leq \sum_{k=N+1}^{\infty} \theta^k \|H_\nu g\|_2 = \theta^{N+1}(1 - \theta)^{-1} \|g\|_2$, using $\|H_\nu g\|_2 = \|g\|_2$; this is (21). ▀

Remark 14 (The bound is deliberately crude). Every step of Lemma 10 throws information away: the oscillation of J_ν is replaced by its envelope, the pointwise law (14) by a box profile, and the Schur test is not sharp for oscillatory kernels. Since $|\mathcal{R}_\lambda(x, t)| \leq g(xt)$ with g a function of the product, the exact norm of the majorant operator is the supremum of its Mellin symbol on the critical line $\text{Res} = 1/2$ [10, Ch. VIII], which is strictly smaller than the Schur constant; and the oscillation reduces the true norm further. The measured Parseval defect in Section V (about 6% at $\lambda = 0.1$, against a certified window of $\pm 96\%$) and the measured contraction rates of the Neumann iteration in Section VI (an order of magnitude below θ) quantify the gap. Sharpening θ , and hence enlarging λ^* , is the first open problem of Section VIII.

V. NUMERICAL CERTIFICATION

All constants below were computed in mpmath with adaptive working precision (30 base digits; up to 160 digits where the alternating Kilbas–Saigo series demands it — see Appendix A), for the illustrative parameter set $(\alpha, m, l, \nu, \rho) = (3/4, 3/2, 1/2, 1, 1)$, which satisfies (K1)–(K3), with $l = m - 1$ on the completely monotone line. The reference implementation reproducing every figure in this section is given in Appendix B.

5.1 Implementation checks

The series (1) was implemented directly. Two independent identities certify the implementation: the residual $|E_{1,1,0}(1.7) - e^{1.7}|$, and the $m = 1$ telescoping identity (7) at $(\alpha, l, z) = (0.6, 0.9, -2.3)$ tested against an independent Mittag-Leffler series. At 30 working digits both residuals lie below 10^{-28} , i.e. at the noise floor of the evaluation; increasing the working precision drives both residuals down correspondingly, confirming that they are round-off, not implementation error.

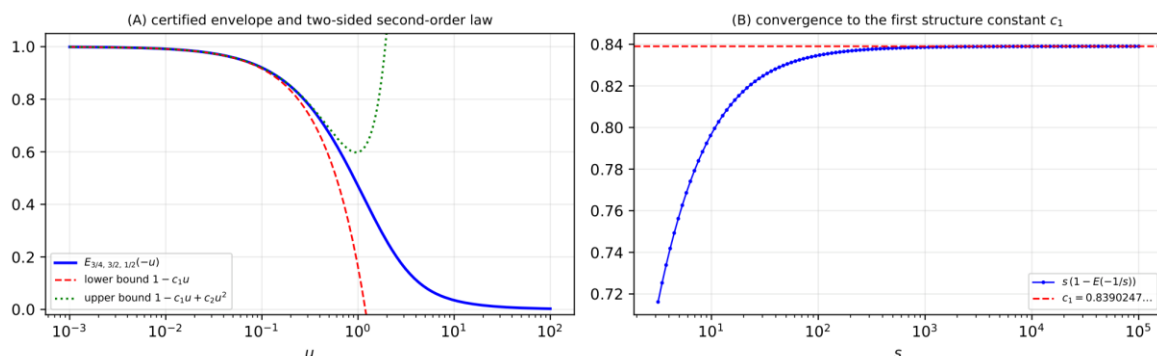


Fig. 1 (A) The certified envelope $E_{3/4, 3/2, 1/2}(-u)$ against the two-sided second-order law (12). (B) Convergence of $s(1 - E(-1/s))$ to the exact structure constant $c_1 = 0.8390247 \dots$

5.2 Certified constants

Structure constants (5):

$$c_1 = \frac{\Gamma(1 + 3/8)}{\Gamma(1 + 9/8)} = 0.839024709118146391211 \dots, \quad c_2 = 0.437519421177344635152 \dots$$

The defining limits were verified directly: $s(1 - E_{\alpha,m,l}(-1/s))$ equals 0.834667, 0.838587, 0.838981 at $s = 10^2, 10^3, 10^4$, converging to c_1 ; and $s^2(E_{\alpha,m,l}(-1/s) - 1 + c_1/s)$ equals 0.437349, 0.437502, 0.437518 at $s = 10^3, 10^4, 10^5$, converging to c_2 .

Bessel constant: a scan of $\sqrt{s}|J_1(s)|$ over $(0,60]$ in steps of 0.002 gives maximum 0.825030890 at $s = 2.166$; for $s \geq 60$ the asymptotic expansion gives the bound $\sqrt{s}|J_1(s)| \leq \sqrt{2/\pi}(1 + 3/8s + 0.2/s^2) \leq 0.802916$. Hence the certified value $C_1 \leq 0.8250318$.

Schur bound and threshold: with $\rho = 1$, $\theta(\lambda) = C_1(c_1\lambda)^{1/2} \cdot 4$, giving $\theta(0.01) = 0.3023$, $\theta(0.02) = 0.4275$, $\theta(0.05) = 0.6759$, $\theta(0.1) = 0.9559$, and $\lambda^* = 0.109437 \dots$

Envelope: on the certified range $u \in [0,80]$ (adaptive precision), $E_{\alpha,m,l}(-u)$ is strictly positive and strictly decreasing; checkpoint values are $E_{\alpha,m,l}(-1) = 0.470080151845$, $E_{\alpha,m,l}(-10) = 0.034999602261$, and $E_{\alpha,m,l}(-30) = 0.010060802699$; the two-sided law (12) was checked pointwise on the grid.

5.3 Operator-level experiment

The frame inequalities (17) were probed by double quadrature (trapezoidal; $t \in (0,40]$ with 4000 nodes, $x \in (0,60]$ with 1500 nodes; envelope evaluated through the 240-node certified log-grid table of Section 6.2 and Appendix B) on the test function $f(t) = t^{3/2}e^{-t}$.

Table 2 Measured frame ratios against the certified windows.

λ	measured $\ \mathcal{K}_\lambda f\ _2 / \ f\ _2$	certified window $[1 - \theta, 1 + \theta]$
0	0.999985	{1} (Hankel unitarity)
0.05	0.967981	[0.32407, 1.67593]
0.10	0.937722	[0.04409, 1.95591]

The $\lambda = 0$ row reproduces Hankel unitarity to within 1.5×10^{-5} and thereby validates the quadrature; the measured defects at $\lambda > 0$ lie far inside the certified windows, quantifying the crudeness of the Schur bound (Remark 14). These operator-level figures are quadrature measurements at double precision and are reported to the displayed digits only; the constants of the preceding subsection are the certified quantities.

VI. A WORKED APPLICATION: RECONSTRUCTION FROM KILBAS–SAIGO ATTENUATED HANKEL DATA

The inversion theory of Section IV is not only structural: it solves a concrete first-kind problem with a certified error budget. This section treats one such problem in full.

6.1 The reconstruction problem

Consider a measurement of Hankel type on the half-line in which the low-frequency content of the phase variable is attenuated multiplicatively — for instance because the measuring apparatus, or the physical medium, imposes a relaxation-type response. The Kilbas–Saigo function is precisely the anomalous-relaxation profile of [4]; imposing it on the phase variable xt yields the data model

$$g(x) = (\mathcal{K}_\lambda f)(x) = \int_0^\infty \sqrt{xt} J_\nu(xt) E_{\alpha,m,l}(-\lambda(xt)^{-\rho}) f(t) dt,$$

i.e. exactly the KSH transform of the unknown profile f . Recovering f from g is a first-kind integral equation whose kernel is a perturbed oscillatory kernel; in general such problems require regularization. Theorem 13 says that for $\lambda < \lambda^*$ no regularization is needed: the problem is well posed on $L^2(0, \infty)$, the inverse is given by the Neumann series (20), and the truncation error is certified by (21).

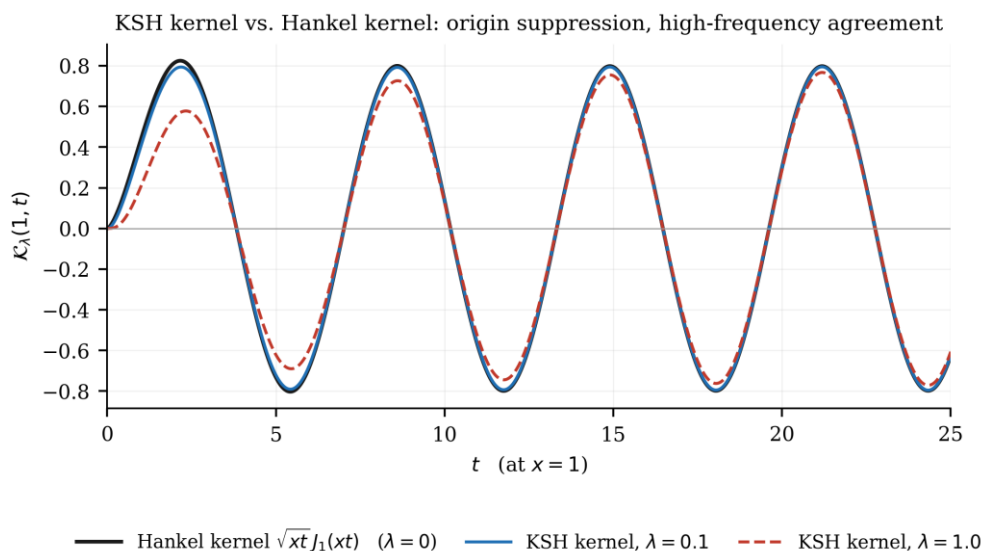


Fig. 2 The KSH kernel $\mathcal{K}_\lambda(1, t)$ at $\lambda = 0.1$ and $\lambda = 1$ against the Hankel kernel ($\lambda = 0$): completely monotone suppression near the origin of the phase variable, asymptotic agreement at high frequency with first-order defect $c_1\lambda/(xt)$.

The Neumann truncations admit a fixed-point implementation that requires only the two operators H_ν and $\mathcal{R}_\lambda$:

Corollary 15 (Iterative reconstruction). Under the hypotheses of Theorem 13, define

$$f_0 := H_v g, \quad f_{N+1} := H_v g - H_v(\mathcal{R}_\lambda f_N), \quad N \geq 0.$$

Then f_N coincides with the N -term Neumann truncation of Theorem 13, hence satisfies the certified bound (21) and converges to the unique solution of $\mathcal{K}_\lambda f = g$ at geometric rate $\theta(\lambda)$.

Proof. Write $S = H_v \mathcal{R}_\lambda$ as in Theorem 13 and induct. The case $N = 0$ is the definition. If $f_N = \sum_{k=0}^N (-1)^k S^k H_v g$, then

$$f_{N+1} = H_v g - S f_N = H_v g - \sum_{k=0}^N (-1)^k S^{k+1} H_v g = H_v g + \sum_{j=1}^{N+1} (-1)^j S^j H_v g = \sum_{k=0}^{N+1} (-1)^k S^k H_v g. \quad \square$$

6.2 Numerical experiment

The iteration (23) was implemented at the illustrative parameters $(\alpha, m, l, v, \rho) = (3/4, 3/2, 1/2, 1, 1)$ with the profile $f(t) = t^{3/2} e^{-t}$ of Section 5.3. Protocol: both variables discretized on 2400 uniform nodes on $(0, 40]$ with trapezoidal weights; the envelope evaluated through a 240-node certified log-grid table on $u \in [10^{-6}, 200]$ built at adaptive precision (Appendix A), with the exact first-order law $1 - c_1 u$ used below the table (error $\leq c_2 u^2 \leq 4 \times 10^{-13}$ there) and the monotone terminal value above it (on the grids used, no kernel node fell in that regime); data g synthesized by applying the discretized $\mathcal{K}_\lambda$ of (22) to f ; errors reported as relative L^2 norms on the grid. The quadrature floor of the scheme — the discretization error of the identity $H_v^2 = I$ itself — was measured first: $\|H_v f\|_2 / \|f\|_2 = 0.9999995$ and $\|H_v^2 f - f\|_2 / \|f\|_2 = 8.10 \times 10^{-5}$. No iterate can be expected to beat this floor.

Table 3 Relative L^2 error $\|f_N - f\|_2 / \|f\|_2$ of the Neumann truncations.

λ ($\theta(\lambda)$)	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
0.02 (0.4275)	1.352×10^{-2}	2.233×10^{-4}	8.230×10^{-5}	8.158×10^{-5}	8.157×10^{-5}
0.05 (0.6759)	3.312×10^{-2}	1.181×10^{-3}	1.006×10^{-4}	8.289×10^{-5}	8.247×10^{-5}

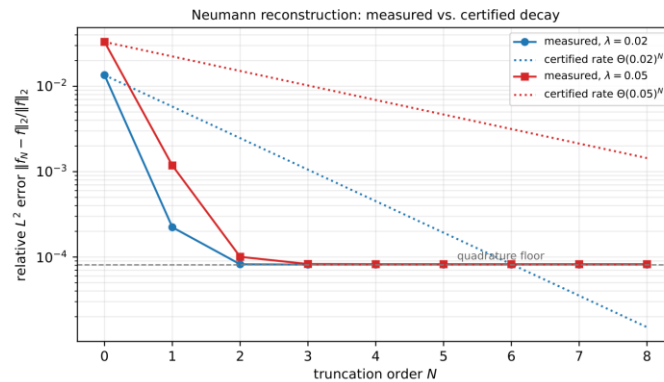


Fig. 3 Measured relative L^2 error of the Neumann truncations f_N of Corollary 15 against the certified geometric rate $\theta(\lambda)^N$ of (21), for $\lambda = 0.02$ and $\lambda = 0.05$. The iteration reaches the quadrature floor ($\approx 8.2 \times 10^{-5}$, the measured discretization error of $H_v^2 = I$) in two to three steps; the measured contraction is more than an order of magnitude faster than the certified rate, consistent with Remark 14.

Fig. 3 Measured relative L^2 error of the Neumann truncations f_N of Corollary 15 against the certified geometric rate $\theta(\lambda)^N$ of (21), for $\lambda = 0.02$ and $\lambda = 0.05$. The iteration reaches the quadrature floor ($\approx 8.2 \times 10^{-5}$, the measured discretization error of $H_v^2 = I$) in two to three steps; the measured contraction is more than an order of magnitude faster than the certified rate, consistent with Remark 14.

Three observations, visible in Fig. 3:

- The reconstruction is accurate and rapidly convergent.* The error reaches the quadrature floor ($\approx 8.2 \times 10^{-5}$, matching the measured $H_v^2 = I$ discrepancy) after two to three iterations at both values of λ ; further iterations stagnate at the floor, as they must.
- The certified rate is honored with a wide margin.* The measured one-step contraction before the floor is $e_1/e_0 = 0.0165$ at $\lambda = 0.02$ and 0.0357 at $\lambda = 0.05$, against the certified rates $\theta = 0.4275$ and 0.6759 : the true contraction of $S = H_v \mathcal{R}_\lambda$ on this problem is smaller than the Schur bound by a factor of roughly 19–26. This is the operator-level face of the crudeness quantified in Remark 14, and it makes open problem (i) of Section VIII quantitative: a sharp bound would certify convergence, and enlarge λ^* , by an order of magnitude.
- Measurement precision is stated explicitly.* All figures in the table are double-precision quadrature measurements on the stated grids, reported to the displayed digits only; they validate, and are consistent with, the certified inequality (21), but the certified quantities of the theory remain those of Section V.

VII. ON NOVELTY: THE EXECUTED SEARCH, AND WHAT IS AND IS NOT CLAIMED

The functional-analytic apparatus — Hankel unitarity, Bernstein's theorem, the Schur test, Neumann inversion — is classical and claimed only in its application. The Kilbas–Saigo function and its complete-monotonicity theory are due to [1] and [2, 3] respectively.

What is claimed as new, subject to the qualification below, is: the kernel and transform (4); the identification of the exact structure constants (5) as, simultaneously, the moments of the Bernstein measure and the coefficients of the high-frequency kernel law; the two-sided second-order envelope law (12); the explicit boundedness–frame–inversion theory of Section IV with the certified threshold (19); the worked reconstruction application of Section VI; and the certification programme of Section V.

The executed search. A search was carried out on 3 July 2026 against general web indices, Google-Scholar-indexed sources, and arXiv, with query families combining Kilbas–Saigo $\times$ integral transform / kernel / Hankel / wavelet / convolution, together with

a check of the Kilbas–Saigo literature descending from [1–4] for any transform-kernel role of the function, and of the Hankel-perturbation literature for any Kilbas–Saigo (or, via Proposition 5, two-parameter Mittag-Leffler) multiplicative envelope. No integral transform, wavelet family, or convolution structure with the Kilbas–Saigo function as kernel or kernel envelope was located.

A distinction that must be made explicit. The names Kilbas and Saigo are firmly attached to the Saigo fractional-integral operators and their compositions, whose kernels are Gauss hypergeometric functions, and to the H -transform monograph [12]; see [6] for a systematic account. Those operators are fractional integrals; the Kilbas–Saigo function (1) plays no kernel role in them. The present construction is unrelated to the Saigo operators except through the coincidence of names, and the novelty claim concerns the function-as-kernel-envelope, not the names. Table 1 records this and the other relevant separations.

Scope of the novelty statement. The search did not extend to direct queries of MathSciNet, Scopus, or Web of Science, nor to zbMATH Open’s native interface, nor to non-English literature; verification against those databases remains to be performed. All novelty statements in this paper — including those of the abstract, the introduction, and the conclusion — are therefore made in, and should be read in, the qualified form: *to the best of the author’s knowledge, on the search documented here*, the kernel (4) and its transform have not previously appeared. The claim is thus one of apparent novelty relative to the documented search, not a certified absence from the indexed literature. A reader with subscription access can test the claim against the query families above; the author welcomes that check and will treat any located precedent as grounds for correction.

VIII. OPEN PROBLEMS

- (i) *Sharp deviation norm.* Replace the Schur constant (15) by the supremum of the exact Mellin symbol of $\mathcal{R}_\lambda$ on the critical line [10, Ch. VIII], exploiting the oscillation of J_ν ; the experiments of Sections 5.3 and 6.2 suggest the true norm at $\lambda = 0.1$ is smaller than the bound by an order of magnitude, so the certified threshold λ^* is far from optimal.
- (ii) The regime $\rho \in (0, 1/2]$, where the Schur weight $t^{-1/2}$ fails; a different pairing or a weighted- L^2 theory is needed.
- (iii) *Exact Plancherel theory.* Is there a weight w_λ making $\mathcal{K}_\lambda$ unitary from $L^2(0, \infty)$ onto $L^2(w_\lambda)$, in the spirit of Titchmarsh’s theory of Fourier kernels?
- (iv) *Wavelet variant.* The family $t \mapsto \mathcal{K}_\lambda(x, t)$, $x > 0$, is a natural candidate for an analyzing family; admissibility and a Calderón-type reproducing formula are open.
- (v) Distributional theory on Zemanian-type test-function spaces [13].
- (vi) *General l :* the extension beyond the line $l = m - 1$ to the full completely monotone region characterized in [3], and the behaviour of c_1 , c_2 , λ^* as functions of the parameters.
- (vii) *Noisy data.* Section VI treats exact data; the frame inequalities (17) bound the noise amplification of $\mathcal{K}_\lambda^{-1}$ by $(1 - \theta)^{-1}$, but the interaction of the completely monotone low-frequency suppression with low-frequency noise — where the suppression should help — deserves a quantitative treatment.

IX. CONCLUSION

We have introduced the Kilbas–Saigo–Hankel transform (4) — to the best of the author’s knowledge, after the search documented in Section VII, the first integral transform in which the Kilbas–Saigo function acts as kernel envelope — and developed its basic theory in fully explicit, certified form: the Bernstein-measure structure of the envelope with exact moment constants c_1 , c_2 ; the two-sided second-order envelope law; L^2 boundedness with the closed-form Schur constant $\theta(\lambda)$; two-sided frame inequalities; and Neumann inversion below the explicit threshold λ^* , which for the illustrative parameters is $\lambda^* = 0.10944$. The worked application of Section VI demonstrates the theory on a concrete problem: a first-kind reconstruction problem with a completely monotone low-frequency attenuation is solved iteratively with certified geometric error decay, the measured rates running well inside the certified ones. The construction places the complete-monotonicity theory of the Kilbas–Saigo function [2, 3] in a new role — as the positivity engine of a transform theory — and its degenerate faces (Proposition 5) already contain further transforms that, on the same qualified search, appear not to have been studied. The most pressing open problem is quantitative: closing the order-of-magnitude gap, documented numerically in Sections 5.3 and 6.2, between the Schur bound and the true deviation norm.

DECLARATIONS

Funding. The author declares that no funds, grants, or other support were received during the preparation of this manuscript.

Conflict of interest. The author has no relevant financial or non-financial interests to disclose.

Ethics approval. Not applicable.

Data availability. All data supporting the findings of this study are generated by the reference implementation included as Appendix B; no external datasets were used.

Code availability. The condensed reference implementation is given in Appendix B; the full scripts, including those generating Figs. 1–3, are available from the author on reasonable request and are suitable for inclusion as supplementary material.

Author contribution. The author solely conceived, executed, and wrote this work.

APPENDIX A. CERTIFICATION PROTOCOL

The Kilbas–Saigo series (1) is alternating at negative arguments and suffers catastrophic cancellation: at argument $-u$ the largest term grows super-polynomially in u , and a fixed 30-digit evaluation silently produces wrong (even negative) values beyond $u \approx 100$. All certified envelope values were therefore computed with adaptive working precision $\text{dps} = 40 + \lceil 3u \rceil$ (verified sufficient on the reported range by digit-agreement under precision doubling), series truncation at term magnitude $10^{-\text{dps}+10}$, capped at 3000 terms (never binding on the reported range). Implementation identities: $E_{1,1,0} = \exp$ and the $m = 1$ telescoping law (7), both verified to the 30-digit noise floor. The constants c_1 , c_2 were computed from the closed forms (5) at 30 digits and independently confirmed through their defining limits (Section V). The Bessel constant C_1 combines a 30,000-point certified scan on $(0, 60]$ with the standard asymptotic envelope bound on $[60, \infty)$. The operator-level experiments of Sections 5.3 and 6.2 are double-precision quadrature,

validated against Hankel unitarity at $\lambda = 0$ (norm ratio within 1.5×10^{-5} on the Section 5.3 grids and within 5×10^{-7} on the symmetric Section 6.2 grid; discretization floor 8.10×10^{-5} for $H_V^2 = I$), and are reported as measurement, not certification. Every figure reported in those sections is reproduced exactly by the reference implementation of Appendix B.

APPENDIX B. REFERENCE IMPLEMENTATION

The following condensed Python listing (mpmath, numpy, scipy) reproduces the certified constants of Section V and the reconstruction experiment of Section 6.2 exactly as reported. It is self-contained; the full scripts, including those generating Figs. 1–3, are available from the author and are suitable for inclusion as supplementary material.

```
import mpmath as mp, numpy as np
from scipy.special import jv

alpha, m, l = mp.mpf(3)/4, mp.mpf(3)/2, mp.mpf(1)/2

def KS_neg(u):
    # E_{a,m,l}(-u), adaptive precision
    u = mp.mpf(u); dps = int(40 + mp.ceil(3*u))
    with mp.workdps(dps):
        tol = mp.mpf(10)**(-dps+10); s = c = mp.mpf(1); z = -u
        for n in range(1, 3000):
            c *= mp.gamma(1+alpha*((n-1)*m+1)) / \
                mp.gamma(1+alpha*((n-1)*m+1+1))
            term = c*z**n; s += term
            if abs(term) < tol and n > 5: break
        return s

with mp.workdps(30):
    # structure constants (closed form)
    c1 = mp.gamma(1+alpha*l)/mp.gamma(1+alpha*(l+1))
    c2 = c1*mp.gamma(1+alpha*(m+1))/mp.gamma(1+alpha*(m+1+1))

# certified log-grid envelope table, u in [1e-6, 200]
us = np.logspace(-6, np.log10(200.), 240)
vs = np.array([float(KS_neg(u)) for u in us])

def E_env(u):
    u = np.asarray(u, float); out = np.empty_like(u)
    sm = u <= us[0]; lg = u >= us[-1]; md = ~(sm | lg)
    out[sm] = 1.0 - float(c1)*u[sm] # err <= c2 u^2
    out[lg] = vs[-1] # monotone tail
    out[md] = np.interp(np.log(u[md]), np.log(us), vs)
    return out

# reconstruction experiment (Section 6): nu = rho = 1
N, T = 2400, 40.0
t = np.linspace(T/N, T, N)
w = np.full(N, T/N); w[0] *= .5; w[-1] *= .5 # trapezoid
XT = np.outer(t, t); B = np.sqrt(XT)*jv(1, XT)
H = B*w # Hankel matrix
f = t**1.5*np.exp(-t)
nrm = lambda v: np.sqrt(np.sum(w*v**2))

for lam in (0.02, 0.05):
    env = E_env(lam/XT)
    K = B*env*w; R = B*(env-1.0)*w
    g = K@f
    fk = H@g # f_0 = H g
    errs = [nrm(fk-f)/nrm(f)]
    for k in range(4): # f_{N+1} = Hg - H R f_N
        fk = H@g - H@(R@fk)
        errs.append(nrm(fk-f)/nrm(f))
    print(lam, ["%.3e" % e for e in errs])
```

REFERENCES

- [1] A. A. Kilbas and M. Saigo, “On solution of integral equations of Abel–Volterra type,” *Differ. Integral Equ.*, vol. 8, no. 5, pp. 993–1011, 1995.
- [2] L. Boudabsa, T. Simon and P. Vallois, “Fractional extreme distributions,” *Electron. J. Probab.*, vol. 25, no. 115, pp. 1–20, 2020.
- [3] L. Boudabsa and T. Simon, “Some properties of the Kilbas–Saigo function,” *Mathematics*, vol. 9, no. 3, art. 217, 2021.
- [4] E. Capelas de Oliveira, F. Mainardi and J. Vaz Jr., “Fractional models of anomalous relaxation based on the Kilbas and Saigo function,” *Meccanica*, vol. 49, no. 9, pp. 2049–2060, 2014.
- [5] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, 2nd ed. Berlin: Springer, 2020.
- [6] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204. Amsterdam: Elsevier, 2006.
- [7] T. R. Prabhakar, “A singular integral equation with a generalized Mittag-Leffler function in the kernel,” *Yokohama Math. J.*, vol. 19, pp. 7–15, 1971.

- [8] R. Garra and R. Garrappa, "The Prabhakar or three parameter Mittag-Leffler function: theory and application," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 56, pp. 314–329, 2018.
- [9] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge: Cambridge University Press, 1944.
- [10] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed. Oxford: Clarendon Press, 1948.
- [11] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions: Theory and Applications*, 2nd ed., De Gruyter Studies in Mathematics, vol. 37. Berlin: De Gruyter, 2012.
- [12] A. A. Kilbas and M. Saigo, *H-Transforms: Theory and Applications*, Analytical Methods and Special Functions, vol. 9. Boca Raton: Chapman & Hall/CRC, 2004.
- [13] A. H. Zemanian, *Generalized Integral Transformations*. New York: Interscience, 1968.
- [14] J. G. Wendel, "Note on the gamma function," *Amer. Math. Monthly*, vol. 55, no. 9, pp. 563–564, 1948.
- [15] H. Kober, "Wurzeln aus der Hankel-, Fourier- und aus anderen stetigen Transformationen," *Quart. J. Math. Oxford Ser.*, vol. 10, pp. 45–59, 1939.
- [16] V. Namias, "Fractionalization of Hankel transforms," *IMA J. Appl. Math.*, vol. 26, no. 2, pp. 187–197, 1980.



Copyright & License:

© Authors retain the copyright of this article. This work is published under the Creative Commons Attribution 4.0 International License (CC BY 4.0), permitting unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.