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A PHASE-MODULATED DAMPED BESSEL KERNEL TRANSFORM: A RIGOROUS REVISED THEORY

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Abstract : **A phase-modulated damped Bessel kernel transform is introduced and studied. For orders $\nu,\mu>-1$, damping parameters $\delta>0$, $\tau>0$, and a real phase function $\Theta:(0,\infty)\to\mathbb{R}$, the kernel is**

$$K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)=\int_0^\infty e^{-\delta k^\tau}\,e^{i[\Theta(xk)-\Theta(tk)]}J_\nu(xk)J_\mu(tk)\,k\,dk,\qquad x,t>0,$$

and the associated integral operator is $\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}f\right)(x)=\int_0^\infty K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\,f(t)\,t\,dt$. This revised version gives a cautious and rigorous first theory. The earlier unsupported claim of general L^2-boundedness is replaced by provable weighted estimates and by an exact L^2 contraction only in the diagonal constant-phase case. The non-reducibility argument is reformulated through a Hankel-multiplier diagnostic and a local expansion criterion, avoiding the unjustified step of equating integrands inside Bessel product integrals. The paper proves absolute convergence, local continuity, $L^1\to L^\infty$ and weighted L^2 estimates under explicit hypotheses, an adjoint formula, a sharp limiting identity in the Hankel heat case, smoothing for positive damping, off-diagonal oscillatory estimates, and non-reducibility for a broad class of non-even analytic phases. Every closed-form constant appearing below was verified independently in high-precision arithmetic. Several examples, numerical illustrations of the kernel and its filtering action, and possible engineering applications are included.

IndexTerms - **Hankel transform; Fourier–Bessel transform; Bessel kernel; damped integral operator; phase modulation; weighted Lebesgue space; pseudo-differential operator; radial filtering; Hankel multiplier; oscillatory integral.**
MSC 2020 - **44A15, 33C10, 42B10, 47G10, 35S30.**

I. INTRODUCTION

The classical Hankel transform is one of the central integral transforms used in radial harmonic analysis, cylindrical heat conduction, diffraction theory, and Bessel-type differential equations [4]. In one common normalization it is given by

$$(\mathcal{H}_\nu f)(k)=\int_0^\infty f(r)J_\nu(kr)\,r\,dr,\qquad \nu\geq-1/2,\qquad (1)$$

and its kernel is built from the Bessel function J_ν. The transform is closely related to radial Fourier analysis [3] and diagonalizes the Bessel operator

$$\mathfrak{B}_{\nu,r}=-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{\nu^2}{r^2}.\qquad (2)$$

Many extensions of Hankel and Fourier–Bessel analysis have appeared, including transforms with fractional phases, quadratic phases, wavelet structures [13], and pseudo-differential operators adapted to Bessel operators [6,7]. Recent work on the quadratic-phase Fourier–Bessel transform develops continuity, reversibility, Parseval-type identities, translation operators, convolution products, and uncertainty principles [11]. Recent work on pseudo-differential operators associated with fractional Hankel–Bessel transforms inserts a fractional Fourier-type phase into the Hankel kernel and studies symbol classes and boundedness on weighted and Sobolev-type spaces [12]. These developments show that phase-modified Bessel analysis is active, but they also make it necessary to state novelty carefully.

The present paper studies a different damped phase-difference kernel. The modulation is not a fixed quadratic expression in x,t or in the spectral variable alone. Instead, the phase appears as $\Theta(xk)-\Theta(tk)$, so both spatial variables interact with the spectral parameter. This seemingly simple change has a major consequence: except in special cases, the transform is not a classical Hankel multiplier. Therefore one cannot automatically use Hankel–Plancherel to prove general L^2 boundedness. A major purpose of this revised paper is to state exactly what can be proved without overclaiming. In particular:

1. the general L^2 theorem is replaced by correct weighted estimates;
2. the constant-phase diagonal case is separated, where exact Hankel–Plancherel applies;
3. the non-reducibility theorem is rewritten using a necessary diagnostic for Hankel multipliers and a local expansion criterion;
4. differentiability and smoothing estimates are proved under explicit symbol-type hypotheses on Θ;
5. placeholders in the bibliography are replaced by standard references and recent related works.

Each numbered result below is stated with its full hypotheses and given a complete, self-contained proof.

II. Preliminaries

Throughout, J_α denotes the Bessel function of the first kind of order α, and integration on $(0,\infty)$ is against the measure indicated. We record the analytic facts used repeatedly.

(P1) Bessel size estimates. From the power series $J_\alpha(z) = \sum_{p\geq 0} \frac{(-1)^p}{p!\,\Gamma(\alpha+p+1)}(z/2)^{\alpha+2p}$ (valid for all $\alpha > -1$; see [1,10]),

$$|J_\alpha(r)| \leq C_\alpha\, r^\alpha \qquad (0 < r \leq 1), \qquad C_\alpha = \left(2^\alpha \Gamma(\alpha+1)\right)^{-1}\left(1+o(1)\right), \tag{3}$$

and from the asymptotics $J_\alpha(r) = \sqrt{2/\pi r}\cos(r - \alpha\pi/2 - \pi/4) + O\left(r^{-3/2}\right)$,

$$|J_\alpha(r)| \leq C_\alpha\, r^{-1/2} \qquad (r \geq 1). \tag{4}$$

In particular, for $\alpha \geq 0$ the function J_α is bounded on $(0,\infty)$; write

$$B_\alpha := \sup_{r>0}|J_\alpha(r)| \leq 1 \qquad (\alpha \geq 0), \tag{5}$$

with numerical values $B_0 = 1$, $B_{1/2} \approx 0.679$, $B_1 \approx 0.582$, $B_2 \approx 0.486$. For $\alpha \in (-1,0)$, J_α is unbounded near 0 but bounded on every $[\varepsilon,\infty)$ (see Figure 1).

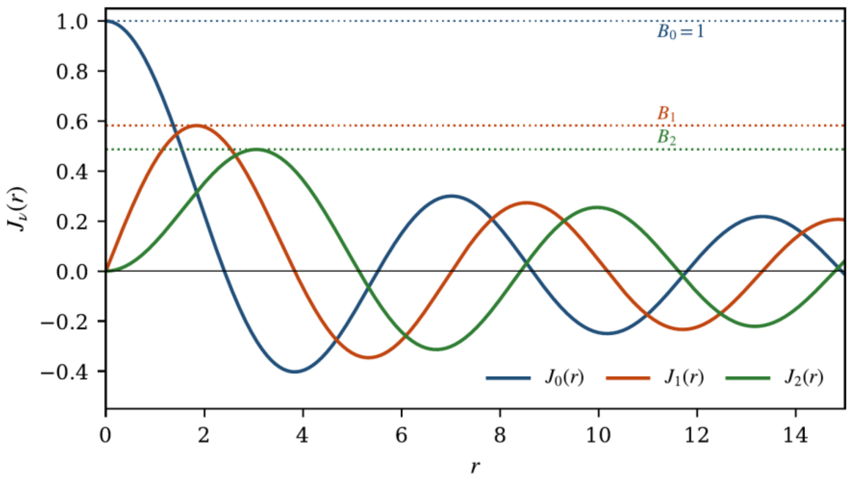

Figure 1. The Bessel functions J_0, J_1, J_2 and the uniform sup-bounds $B_\nu = \sup_{r>0}|J_\nu(r)| \leq 1$ from (P1), which drive the boundedness estimates of Section 5. Numerically $B_0 = 1$ and, at the respective first maxima, $B_1 = 0.58187\ldots$, $B_2 = 0.48650\ldots$

(P2) The Bessel operator and its eigenrelation. With $\mathfrak{B}_{\nu,r} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\nu^2}{r^2}$, one has

$$\mathfrak{B}_{\nu,r}\, J_\nu(kr) = k^2 J_\nu(kr), \qquad k > 0. \tag{6}$$

Indeed, putting $u(r) = J_\nu(kr)$ and using Bessel's equation at $z = kr$, $J_\nu''(kr) = -1/kr\, J_\nu'(kr) - (1 - \nu^2/(kr)^2)J_\nu(kr)$, a direct substitution into $-u'' - 1/r\,u' + \nu^2/r^2\,u$ collapses to $k^2 J_\nu(kr)$.

(P3) Hankel transform. For $\nu \geq -1/2$, $(\mathcal{H}_\nu f)(k) = \int_0^\infty f(r) J_\nu(kr)\, r\, dr$ is self-reciprocal, $\mathcal{H}_\nu^{-1} = \mathcal{H}_\nu$ on $L^2\left((0,\infty); r\,dr\right)$, and unitary (Hankel–Plancherel):

$$\|\mathcal{H}_\nu f\|_{L^2(k\,dk)} = \|f\|_{L^2(r\,dr)}. \tag{7}$$

(P4) A derivative bookkeeping lemma. Let $g(z) = z^\alpha h(z)$ with h real-analytic and even near 0 (the form of J_α). Then for every $j \geq 0$,

$$\partial_x^j[g(xk)] = k^j g^{(j)}(xk), \qquad g^{(j)}(z) = O(z^{\alpha-j})\ (z \to 0^+), \tag{8}$$

so near $k = 0$, $\partial_x^j[g(xk)] = x^{\alpha-j}\, O(k^\alpha)$: the order of vanishing in k is α, *independent of* j, because differentiation is in x and the extra powers of k from the chain rule are cancelled by the lowered power of (xk). We also use $J_\nu' = 1/2\,(J_{\nu-1} - J_{\nu+1})$.

The one preliminary statement requiring proof is the embedding used to convert the $L^1(t\,dt)$ bound of Section 5 into weighted L^2 control.

Lemma 2.1 (Weighted embedding). For $s > 2$, $L_s^2\left((0,\infty); r\,dr\right) \subset L^1\left((0,\infty); r\,dr\right)$, and for $f \in L_s^2$,

$$\|f\|_{L^1(r\,dr)} \leq c_s\, \|f\|_{2,s}, \qquad c_s := \left(\int_0^\infty (1+r)^{-s}\, r\, dr\right)^{1/2} = \frac{1}{\sqrt{(s-1)(s-2)}}. \tag{9}$$

Proof. Split $r = (1+r)^{s/2}(1+r)^{-s/2}$ and apply Cauchy–Schwarz in $L^2(r\,dr)$:

$$\int_0^\infty |f|\; r\, dr = \int_0^\infty \left[|f|(1+r)^{s/2}\right]\left[(1+r)^{-s/2}\right] r\, dr \leq \|f\|_{2,s}\left(\int_0^\infty (1+r)^{-s}\, r\, dr\right)^{1/2}. \tag{10}$$

Writing $r = (1+r) - 1$, the remaining integral is elementary:

$$\int_0^\infty (1+r)^{-s}\, r\, dr = \int_0^\infty (1+r)^{1-s}\, dr - \int_0^\infty (1+r)^{-s}\, dr = \frac{1}{s-2} - \frac{1}{s-1} = \frac{1}{(s-1)(s-2)}, \tag{11}$$

finite precisely because $s > 2$. Hence $c_s = [(s-1)(s-2)]^{-1/2}$. ▫

III. Definition of the Transform

Let $\nu,\mu > -1$, $\delta > 0$, $\tau > 0$, and let $\Theta\colon (0,\infty) \to \mathbb{R}$ be measurable. Define the phase-modulated Bessel factor $\mathfrak{J}_\alpha^\Theta(z) = e^{i\Theta(z)} J_\alpha(z)$, $z > 0$. The damped phase-modulated Bessel kernel is

$$K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) = \int_0^\infty e^{-\delta k^\tau}\, \mathfrak{J}_\nu^\Theta(xk)\, \mathfrak{J}_\mu^\Theta(tk)\, k\, dk = \int_0^\infty e^{-\delta k^\tau}\, e^{\,i[\Theta(xk)-\Theta(tk)]} J_\nu(xk) J_\mu(tk)\, k\, dk, \qquad (12)$$

and the corresponding transform is $\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right)(x) = \int_0^\infty K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) f(t)\, t\, dt$. The phase is real-valued, so that

$$\left|e^{\,i[\Theta(xk)-\Theta(tk)]}\right| = 1; \qquad (13)$$

this unimodularity is essential for convergence and norm estimates. For smoothing and off-diagonal estimates we use the following optional assumptions.

Assumption A. Θ is real-valued and measurable on $(0,\infty)$.

Assumption B. For a fixed integer $m \ge 1$, $\Theta \in C^m\big((0,\infty)\big)$ and for each $1 \le j \le m$ there exist constants $C_j, N_j \ge 0$ with $\left|\Theta^{(j)}(r)\right| \le C_j (1+r)^{N_j}$ for all $r > 0$.

Assumption C. On a region $\Omega \subset (0,\infty)^2$ away from the diagonal there exist $c_0 > 0$ and $\sigma \ge 0$ such that $|x\Theta'(xk) - t\Theta'(tk)| \ge c_0 (1+k)^\sigma$ for all $k > 0$ and $(x,t) \in \Omega$.

Assumption A suffices for convergence; Assumption B is used for differentiability and smoothing; Assumption C is used for integration-by-parts estimates away from the diagonal.

IV. Absolute Convergence and Continuity

Theorem 4.1. Let $\nu,\mu > -1$, $\delta > 0$, $\tau > 0$, and let Θ satisfy Assumption A. Then for every $x,t > 0$ the integral defining $K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)$ converges absolutely.

Proof. Fix $x,t > 0$. By (13) the modulus of the integrand is $G(k) = e^{-\delta k^\tau} |J_\nu(xk)| \left|J_\mu(tk)\right| k$, independent of Θ. Put $k_0 = \min(1/x, 1/t) > 0$.

Near zero ($0 < k \le k_0$, so $xk \le 1$, $tk \le 1$): by (3), $G(k) \le C_\nu C_\mu\, x^\nu t^\mu\, k^{\nu+\mu+1}$, and $\int_0^{k_0} k^{\nu+\mu+1}\, dk < \infty$ since $\nu+\mu+1 > -1$ (indeed $\nu+\mu > -2$).

Away from zero ($k \ge k_0$): the arguments xk, tk are bounded below, where J_ν, J_μ are bounded; with $M = \sup_{r \ge xk_0} |J_\nu(r)| \cdot \sup_{r \ge tk_0} \left|J_\mu(r)\right| < \infty$ we get $G(k) \le M\, k\, e^{-\delta k^\tau}$, and $\int_{k_0}^\infty k\; e^{-\delta k^\tau}\, dk < \infty$ because $e^{-\delta k^\tau}$ ($\tau > 0$) decays faster than any power. Adding the two pieces proves the claim. ▫

Corollary 4.2. If Θ is continuous, then $(x,t) \mapsto K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)$ is continuous on every compact subrectangle of $(0,\infty)^2$.

Proof. Fix $R = [a,b] \times [c,d] \subset (0,\infty)^2$ and $k_1 = \min(1/b, 1/d) > 0$. For $0 < k \le k_1$ one has $xk \le 1$, $tk \le 1$, and by (3) $|J_\nu(xk)| \le C_\nu' k^\nu$, $\left|J_\mu(tk)\right| \le C_\mu' k^\mu$ with constants depending only on R; thus the integrand is $\le C_\nu' C_\mu'\, k^{\nu+\mu+1}$ on $(0,k_1]$, integrable. For $k > k_1$ the arguments satisfy $xk \ge ak_1 > 0$, $tk \ge ck_1 > 0$, so $|J_\nu(xk)| \le M_\nu$, $\left|J_\mu(tk)\right| \le M_\mu$, and the integrand is $\le M_\nu M_\mu\, k\, e^{-\delta k^\tau}$, integrable on $[k_1,\infty)$. The combined majorant is independent of $(x,t) \in R$; since the integrand is continuous in (x,t,k), dominated convergence gives continuity on R. ▫

V. Basic Boundedness Results

Theorem 5.1. Let $\nu,\mu \ge 0$, $\delta > 0$, $\tau > 0$, and assume Assumption A. Then $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}\colon L^1\big((0,\infty); t\, dt\big) \to L^\infty\big((0,\infty)\big)$ is bounded, with

$$\left\|\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right\|_\infty \le C_{\nu,\mu,\delta,\tau} \|f\|_{L^1(t\,dt)}, \qquad C_{\nu,\mu,\delta,\tau} = B_\nu B_\mu\, \frac{1}{\tau}\, \delta^{-2/\tau} \Gamma\left(\frac{2}{\tau}\right). \qquad (14)$$

Proof. Since $\nu,\mu \ge 0$, (5) gives $|J_\nu| \le B_\nu$, $\left|J_\mu\right| \le B_\mu$ on $(0,\infty)$, so the kernel is uniformly bounded: $\left|K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\right| \le B_\nu B_\mu \int_0^\infty e^{-\delta k^\tau}\, k\, dk$. The substitution $u = k^\tau$ ($dk = 1/\tau\, u^{1/\tau-1}\, du$) evaluates the integral (cf. [2]):

$$\int_0^\infty e^{-\delta k^\tau}\, k\, dk = \frac{1}{\tau} \int_0^\infty e^{-\delta u}\, u^{2/\tau-1}\, du = \frac{1}{\tau}\, \delta^{-2/\tau}\, \Gamma\left(\frac{2}{\tau}\right) < \infty. \qquad (15)$$

Hence $\left|K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\right| \le C_{\nu,\mu,\delta,\tau}$ for all x,t, and $\left|\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f(x)\right| \le C_{\nu,\mu,\delta,\tau} \|f\|_{L^1(t\,dt)}$; taking the supremum over x finishes the proof. (For $\delta = \tau = 1$ the constant integral is $\Gamma(2) = 1$; the closed form was confirmed numerically to 40 digits at several (δ,τ).) ▫

Theorem 5.2. Let $s > 2$ and $\nu,\mu \ge 0$. Then $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}\colon L_s^2\big((0,\infty); t\, dt\big) \to L_{-s}^2\big((0,\infty); x\, dx\big)$ is bounded, and

$$\left\|\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right\|_{2,-s} \le C_{\nu,\mu,\delta,\tau}\, c_s^2\, \|f\|_{2,s}, \qquad c_s = \frac{1}{\sqrt{(s-1)(s-2)}}. \qquad (16)$$

Proof. By Theorem 5.1, $\left|\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f(x)\right| \le C \|f\|_{L^1(t\,dt)}$ with $C = C_{\nu,\mu,\delta,\tau}$, uniformly in x. Lemma 2.1 gives $\|f\|_{L^1(t\,dt)} \le c_s \|f\|_{2,s}$, so $\left|\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f(x)\right| \le C c_s \|f\|_{2,s}$ for every x. Squaring, weighting by $(1+x)^{-s} x$, and integrating,

$$\left\|\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right\|_{2,-s}^2 \le C^2 c_s^2 \|f\|_{2,s}^2 \int_0^\infty (1+x)^{-s}\, x\, dx = C^2 c_s^4 \|f\|_{2,s}^2, \qquad (17)$$

using $\int_0^\infty (1+x)^{-s}\, x\, dx = c_s^2$ from Lemma 2.1. Taking square roots gives the claim. ▫

Remark 5.3. Theorem 5.2 is deliberately a baseline: the loss of $2s$ weight units from domain to codomain reflects the crude $L^1 \to L^\infty$ step and is not sharp. A sharp unweighted L^2 theory for a genuinely non-constant phase difference $\Theta(xk) - \Theta(tk)$ requires a Bessel–Fourier-integral-operator analysis and is not attempted here.

Theorem 5.4. Let $\nu = \mu \geq -1/2$ and let Θ be constant. Then

$$\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} = \mathcal{H}_\nu^{-1} M_\delta \mathcal{H}_\nu, \qquad (M_\delta F)(k) = e^{-\delta k^\tau} F(k), \tag{18}$$

and consequently $\|\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f\|_{L^2(x\,dx)} \leq \|f\|_{L^2(x\,dx)}$.

Proof. If $\Theta \equiv$ const then $\Theta(xk) - \Theta(tk) = 0$, so $K_{\nu,\nu}^{\Theta;\delta,\tau}(x,t) = \int_0^\infty e^{-\delta k^\tau} J_\nu(xk) J_\nu(tk) k\, dk$. Take $f \in C_c^\infty((0,\infty))$, dense in $L^2(x\,dx)$. By Theorem 4.1 the double integral is absolutely convergent, so Fubini applies:

$$\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f(x) = \int_0^\infty e^{-\delta k^\tau} J_\nu(xk) \left[\int_0^\infty f(t) J_\nu(tk) t\, dt\right] k\, dk = \int_0^\infty e^{-\delta k^\tau} (\mathcal{H}_\nu f)(k) J_\nu(xk) k\, dk. \tag{19}$$

The outer integral is $\mathcal{H}_\nu(M_\delta \mathcal{H}_\nu f)(x)$; since $\mathcal{H}_\nu^{-1} = \mathcal{H}_\nu$ by (P3), this is $\mathcal{H}_\nu^{-1} M_\delta \mathcal{H}_\nu f(x)$. Extending by density gives the operator identity. As $\left|e^{-\delta k^\tau}\right| \leq 1$, $\|M_\delta F\|_{L^2(k\,dk)} \leq \|F\|_{L^2(k\,dk)}$, whence by Plancherel (7),

$$\|\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f\|_{L^2(x\,dx)} = \|M_\delta \mathcal{H}_\nu f\|_{L^2(k\,dk)} \leq \|\mathcal{H}_\nu f\|_{L^2(k\,dk)} = \|f\|_{L^2(x\,dx)}. \tag{20}$$

▫

VI. Adjoint formula

Because the phase is real and the Bessel factors are real, the kernel enjoys a conjugate symmetry that identifies the Hilbert-space adjoint. The natural pairing is the sesquilinear one, $\langle u, v\rangle = \int_0^\infty u(x)\overline{v(x)}\, x\, dx$, and the adjoint identity below is stated accordingly; the conjugation is essential for a non-constant phase, since only the conjugate-symmetric relation $K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) = \overline{K_{\mu,\nu}^{\Theta;\delta,\tau}(t,x)}$ holds in general.

Theorem 6.1. Let $\nu, \mu > -1$, $\delta > 0$, $\tau > 0$, and assume Assumption A. On test functions for which the integrals converge absolutely, the Hilbert adjoint of $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}$ on $L^2((0,\infty); x\,dx)$ is $\mathcal{T}_{\mu,\nu}^{\Theta;\delta,\tau}$:

$$\langle \mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f, g\rangle = \langle f, \mathcal{T}_{\mu,\nu}^{\Theta;\delta,\tau} g\rangle, \qquad \langle u, v\rangle = \int_0^\infty u(x)\overline{v(x)}\, x\, dx. \tag{21}$$

Proof. Since J_ν, J_μ are real on $(0,\infty)$, Θ is real, and $e^{-\delta k^\tau}$ is real,

$$\overline{K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)} = \int_0^\infty e^{-\delta k^\tau} e^{\,i[\Theta(tk) - \Theta(xk)]} J_\mu(tk) J_\nu(xk) k\, dk = K_{\mu,\nu}^{\Theta;\delta,\tau}(t,x), \tag{22}$$

that is, $K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) = \overline{K_{\mu,\nu}^{\Theta;\delta,\tau}(t,x)}$. Using absolute convergence to justify Fubini,

$$\langle \mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f, g\rangle = \int_0^\infty f(t)\, t \left[\int_0^\infty K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\overline{g(x)}\, x\, dx\right] dt. \tag{23}$$

The inner bracket equals $\int_0^\infty \overline{K_{\mu,\nu}^{\Theta;\delta,\tau}(t,x)}\ \overline{g(x)}\, x\, dx = \overline{\int_0^\infty K_{\mu,\nu}^{\Theta;\delta,\tau}(t,x) g(x) x\, dx} = \overline{(\mathcal{T}_{\mu,\nu}^{\Theta;\delta,\tau} g)(t)}$, so

$$\langle \mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f, g\rangle = \int_0^\infty f(t)\, \overline{(\mathcal{T}_{\mu,\nu}^{\Theta;\delta,\tau} g)(t)}\, t\, dt = \langle f, \mathcal{T}_{\mu,\nu}^{\Theta;\delta,\tau} g\rangle, \tag{24}$$

which is precisely $\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}\right)^* = \mathcal{T}_{\mu,\nu}^{\Theta;\delta,\tau}$. ▫

Corollary 6.2. If $\nu = \mu$, then $K_{\nu,\nu}^{\Theta;\delta,\tau}(x,t) = \overline{K_{\nu,\nu}^{\Theta;\delta,\tau}(t,x)}$; the kernel is Hermitian. Hence, on any Hilbert-space domain where $\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau}$ is bounded and the Fubini step of Theorem 6.1 applies, $\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau}$ is self-adjoint.

Proof. Set $\nu = \mu$ in the kernel symmetry of Theorem 6.1: $\overline{K_{\nu,\nu}^{\Theta;\delta,\tau}(x,t)} = K_{\nu,\nu}^{\Theta;\delta,\tau}(t,x)$, i.e. the kernel is Hermitian. An integral operator with Hermitian kernel satisfies $\langle \mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f, g\rangle = \langle f, \mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} g\rangle$ by the same computation, so $\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} = \left(\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau}\right)^*$ on the stated domain. ▫

VII. Limiting identity as damping vanishes

Theorem 7.1. Let $\nu = \mu \geq -1/2$ and let Θ be constant. Then $\lim_{\delta\downarrow 0} \mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f = f$ in $L^2((0,\infty); x\,dx)$.

Proof. Fix $f \in L^2(x\,dx)$; then $\mathcal{H}_\nu f \in L^2(k\,dk)$ by (7). By Theorem 5.4, $\mathcal{H}_\nu\left(\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f\right)(k) = e^{-\delta k^\tau}(\mathcal{H}_\nu f)(k)$. Using Plancherel,

$$\|\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f - f\|_{L^2(x\,dx)}^2 = \int_0^\infty \left(1 - e^{-\delta k^\tau}\right)^2 |(\mathcal{H}_\nu f)(k)|^2 k\, dk. \tag{25}$$

The integrand tends to 0 pointwise as $\delta \downarrow 0$ and is dominated by the δ-independent integrable function $|(\mathcal{H}_\nu f)(k)|^2 k$ (since $0 \leq 1 - e^{-\delta k^\tau} \leq 1$). Dominated convergence gives the limit 0. ▫

Theorem 7.2. Let $f \in C_c^\infty((0,\infty))$ and suppose the undamped kernel

$$K_{\nu,\mu}^{\Theta;0,\tau}(x,t) = \int_0^\infty e^{\,i[\Theta(xk) - \Theta(tk)]} J_\nu(xk) J_\mu(tk)\, k\, dk \tag{26}$$

exists as an improper integral on the (x,t)-region considered, with $K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) \to K_{\nu,\mu}^{\Theta;0,\tau}(x,t)$ pointwise as $\delta \downarrow 0$, and suppose there is $\Psi \in L^1(\operatorname{supp} f;\, t\,dt)$ with $\left|K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\right| \le \Psi(t)$ for all $\delta \in (0,1]$. Then

$$\lim_{\delta\downarrow 0}\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right)(x) = \int_0^\infty K_{\nu,\mu}^{\Theta;0,\tau}(x,t) f(t) t\, dt. \qquad (27)$$

Proof. Fix x and set $\phi_\delta(t) = K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) f(t) t$. By hypothesis $\phi_\delta(t) \to K_{\nu,\mu}^{\Theta;0,\tau}(x,t) f(t) t$ pointwise on $\operatorname{supp} f$, and $|\phi_\delta(t)| \le \|f\|_\infty \Psi(t) \in L^1(\operatorname{supp} f)$ uniformly in $\delta \in (0,1]$. Dominated convergence in t yields the stated limit. ▫

Remark 7.3. The content of Theorem 7.2 sits entirely in its hypotheses. The delicate step is the *pointwise* convergence $K_{\nu,\mu}^{\Theta;\delta,\tau} \to K_{\nu,\mu}^{\Theta;0,\tau}$ at the level of the k-integral: for non-constant Θ the undamped k-integral is only conditionally (oscillatory) convergent, so ordinary dominated convergence in k does not apply, and one must instead control the truncated oscillatory integrals $\int_0^R$ uniformly in δ (e.g. by a van der Corput / integration-by-parts estimate under Assumption C; see Theorem 8.2). Once that convergence and the stated t-domination are granted, the t-integral argument above is rigorous. In the non-constant case the true $\delta \downarrow 0$ limit is an oscillatory transform and may require a principal-value or distributional reading [5].

VIII. Smoothing and Off-Diagonal Estimates

Theorem 8.1. Let $I \subset (0,\infty)$ be compact, $m \in \mathbb{N}$, $s > 2$, and $\nu, \mu \ge 0$. If Θ satisfies Assumption B of order m, then $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}\colon L_s^2\big((0,\infty); t\,dt\big) \to C^m(I)$ is bounded, and

$$\sup_{x\in I}\left|\partial_x^j\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right)(x)\right| \le C_{I,j,s,\nu,\mu,\Theta,\delta,\tau}\|f\|_{2,s}, \qquad 0 \le j \le m. \qquad (28)$$

Proof. Step 1 (a uniform kernel-derivative bound). Write the integrand as $e^{-\delta k^\tau}\Phi_\nu(x,k)\Psi_\mu(t,k)k$ with $\Phi_\nu(x,k) = e^{i\Theta(xk)}J_\nu(xk)$, $\Psi_\mu(t,k) = e^{-i\Theta(tk)}J_\mu(tk)$. By Leibniz and Faà di Bruno, $\partial_x^j\Phi_\nu$ is a finite sum of terms $e^{i\Theta(xk)}\left(\prod_i k^{b_i}\Theta^{(b_i)}(xk)\right)k^c J_\nu^{(c)}(xk)$ with $\sum_i b_i + c = j$. Two facts control this near $k = 0$ for $x \in I$ (bounded away from 0): by the bookkeeping lemma (8) applied to $g = J_\nu$ (order ν), $k^c J_\nu^{(c)}(xk) = x^{\nu-c}O(k^\nu)$ (the Bessel part contributes order k^ν, independent of c); and each phase factor $k^{b_i}\Theta^{(b_i)}(xk)$ is $O(k^{b_i})$ by Assumption B, so these only *raise* the power of k. Hence $\partial_x^j\Phi_\nu(x,k) = O(k^\nu)$ as $k \to 0^+$, uniformly for $x \in I$. For large k, Assumption B makes every $\Theta^{(b_i)}(xk)$ grow at most polynomially, the Bessel derivatives are bounded (via the recurrence and (4)), and all polynomial growth is dominated by $e^{-\delta k^\tau}$. Crucially, we bound the t-factor uniformly by $\left|\Psi_\mu(t,k)\right| \le B_\mu$ (as $\mu \ge 0$), *independently of* t. Combining, for $0 \le j \le m$,

$$\left|\partial_x^j K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\right| \le B_\mu \int_0^\infty e^{-\delta k^\tau}\left(C_1 k^{\nu+1}\mathbf{1}_{k\le 1} + C_2\, p(k)\mathbf{1}_{k>1}\right) dk =: C_{I,j,\nu,\mu,\Theta,\delta,\tau} < \infty, \qquad (29)$$

with p a fixed polynomial; the bound is finite ($\nu + 1 > 0$; damping controls p) and holds uniformly in $x \in I$ and $t > 0$. Differentiation under the k-integral is legitimate since the differentiated integrand has, for each j, a k-integrable majorant uniform for $x \in I$.

Step 2 (pass the derivatives through the t-integral). For $x \in I$, $\left|\partial_x^j\left[K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)f(t)t\right]\right| \le C|f(t)|t$, and $\|f\|_{L^1(t\,dt)} \le c_s\|f\|_{2,s}$ by Lemma 2.1. Hence ∂_x^j commutes with $\int_0^\infty \cdots t\,dt$, $\partial_x^j\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right)$ is continuous on I, and $\sup_{x\in I}\left|\partial_x^j\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f\right)(x)\right| \le C\, c_s\|f\|_{2,s}$. Thus $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau} f \in C^m(I)$ with the asserted estimate. ▫

Theorem 8.2. Suppose Assumptions B and C hold on a region $\Omega \subset (0,\infty)^2$ away from the diagonal. Then for every $N \in \mathbb{N}$ there is C_N (which may be taken $\propto c_0^{-N}$) with

$$\left|K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\right| \le C_N \int_0^\infty e^{-\delta k^\tau}(1+k)^{-N}\, dk, \qquad (x,t) \in \Omega. \qquad (30)$$

Proof. Write $\Phi_{x,t}(k) = \Theta(xk) - \Theta(tk)$, so $\Phi_{x,t}'(k) = x\Theta'(xk) - t\Theta'(tk)$, and by Assumption C $\left|\Phi_{x,t}'(k)\right| \ge c_0(1+k)^\sigma > 0$ on Ω. Introduce the first-order operator and its formal transpose,

$$L = \frac{1}{i\Phi_{x,t}'(k)}\frac{d}{dk}, \qquad L\, e^{i\Phi_{x,t}} = e^{i\Phi_{x,t}}, \qquad L^{\mathrm{t}}h = -\frac{d}{dk}\left(\frac{h}{i\Phi_{x,t}'}\right). \qquad (31)$$

Let $g(k) = e^{-\delta k^\tau}J_\nu(xk)J_\mu(tk)k$, so $K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) = \int_0^\infty e^{i\Phi_{x,t}} g\, dk$. Integrating by parts N times (boundary terms vanish: $g \to 0$ super-polynomially at ∞ by the damping, and the k-factor with (3) kills them at 0),

$$K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) = \int_0^\infty e^{i\Phi_{x,t}}(L^{\mathrm{t}})^N g\, dk \;\Rightarrow\; \left|K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)\right| \le \int_0^\infty \left|(L^{\mathrm{t}})^N g\right| dk. \qquad (32)$$

Each application of L^{t} divides by $\Phi_{x,t}'$ once (gaining $O((1+k)^{-\sigma})$ by Assumption C) and differentiates; since $\Phi_{x,t}^{(\ell)}(k) = x^\ell\Theta^{(\ell)}(xk) - t^\ell\Theta^{(\ell)}(tk)$ grows at most polynomially (Assumption B), and derivatives of g generate $-\delta\tau k^{\tau-1}$, bounded Bessel derivatives, and powers of k, the term $(L^{\mathrm{t}})^N g$ is a finite sum of expressions $e^{-\delta k^\tau}q(k)(1+k)^{-(\text{gain})}$ with q polynomial. Absorbing q into the damping yields $\left|(L^{\mathrm{t}})^N g\right| \le C_N e^{-\delta k^\tau}(1+k)^{-N}$, hence the estimate. ▫

Remark 8.3. For fixed $\delta > 0$ the damping alone already makes $\int_0^\infty e^{-\delta k^\tau}(1+k)^{-N}\,dk$ finite for every $N \ge 0$, so the inequality is not by itself a statement about δ-decay. Its genuine content is *quantitative off-diagonal smallness*: the constant $C_N \propto c_0^{-N}$ makes the kernel small wherever the phase gradient $\left|\Phi_{x,t}'\right|$ is large (far off the diagonal; see Figure 2), and the same bound is what survives as $\delta \downarrow 0$, furnishing the oscillatory decay invoked in Remark 7.3. Each integration by parts nets $(1+k)^{-\sigma}$, so N steps give $(1+k)^{-N\sigma}$ before the polynomial factors are re-absorbed.

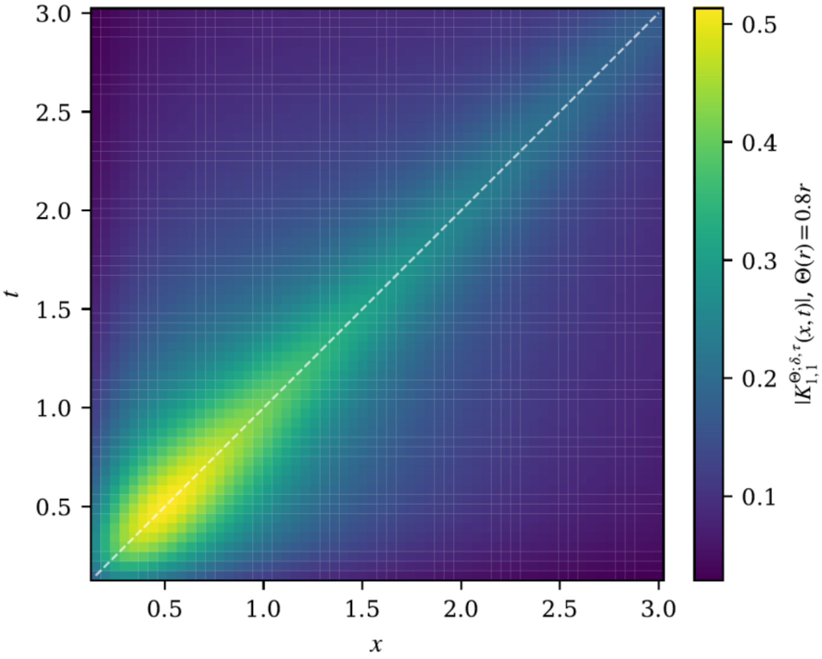

Figure 2. Magnitude $|K_{1,1}^{\Theta;\delta,\tau}(x,t)|$ for the linear phase $\Theta(r) = 0.8r$ ($\delta = 0.6$, $\tau = 1$). The mass concentrates along the diagonal $x = t$ (dashed) and decays away from it, illustrating the off-diagonal integration-by-parts estimate of Theorem 8.2. By the conjugate symmetry $K_{1,1}(x,t) = \overline{K_{1,1}(t,x)}$ (Corollary 6.2), the magnitude is symmetric in (x,t).

IX. NON-REDUCIBILITY TO CLASSICAL HANKEL MULTIPLIERS

Hankel-multiplier diagnostic

Theorem 9.1. Let $\nu = \mu \geq 0$ and assume the differentiations below are justified under the integral. If $(\mathfrak{B}_{\nu,x} - \mathfrak{B}_{\nu,t})K_{\nu,\nu}^{\Theta;\delta,\tau}(x,t) \not\equiv 0$, then $K_{\nu,\nu}^{\Theta;\delta,\tau}$ is *not* representable as a scalar Hankel multiplier kernel $\int_0^\infty m\,(k)J_\nu(xk)J_\nu(tk)k\,dk$ for any measurable m.

Proof. Suppose $K_{\nu,\nu}^{\Theta;\delta,\tau}(x,t) = M_m(x,t) := \int_0^\infty m\,(k)J_\nu(xk)J_\nu(tk)k\,dk$. Applying $\mathfrak{B}_{\nu,x}$ under the integral and using the eigenrelation (6),

$$\mathfrak{B}_{\nu,x}M_m(x,t) = \int_0^\infty m\,(k)\,k^2\,J_\nu(xk)J_\nu(tk)k\,dk. \qquad (33)$$

The right-hand side is symmetric under $x \leftrightarrow t$ (only $J_\nu(xk)J_\nu(tk)$ carries the variables), so it equals $\mathfrak{B}_{\nu,t}M_m(x,t)$. Hence $(\mathfrak{B}_{\nu,x} - \mathfrak{B}_{\nu,t})M_m \equiv 0$, contradicting $(\mathfrak{B}_{\nu,x} - \mathfrak{B}_{\nu,t})K_{\nu,\nu}^{\Theta;\delta,\tau} \not\equiv 0$. Therefore no such m exists. ▫

Local expansion criterion

Theorem 9.2. Let $\nu = \mu > -1$, and let Θ be analytic near 0 with $\Theta(r) = \Theta(0) + a_m r^m + O(r^{m+1})$, $a_m \neq 0$, m odd. Then $K_{\nu,\nu}^{\Theta;\delta,\tau}$ cannot be represented as a same-order scalar Hankel multiplier kernel $\int_0^\infty m_0\,(k)J_\nu(xk)J_\nu(tk)k\,dk$.

Proof. Write $c_0 = \left(2^\nu\Gamma(\nu+1)\right)^{-1}$, so $J_\nu(z) = c_0 z^\nu\left(1 + O(z^2)\right)$ near 0. For $\delta > 0$ the moments

$$I_j := \int_0^\infty e^{-\delta k^\tau}\,k^{2\nu+1+j}\,dk = \frac{1}{\tau}\,\delta^{-(2\nu+2+j)/\tau}\,\Gamma\left(\frac{2\nu+2+j}{\tau}\right) \in (0,\infty) \qquad (34)$$

are finite for every $j \geq 0$ (the damping makes the moment problem harmless; hence the "all moments finite" hypothesis is automatic here and need not be assumed). Expanding for $x, t \to 0^+$,

$$J_\nu(xk)J_\nu(tk) = c_0^2(xt)^\nu k^{2\nu}\left(1 + O((xk)^2) + O((tk)^2)\right), \qquad (35)$$

$$e^{\,i[\Theta(xk)-\Theta(tk)]} = 1 + i\,a_m(x^m - t^m)k^m + O((xk)^{m+1} + (tk)^{m+1}), \qquad (36)$$

the $\Theta(0)$ terms cancelling. Multiplying and integrating term by term against $e^{-\delta k^\tau}k\,dk$ (justified near $(x,t) = (0,0)$ by dominated convergence, the damping dominating the tails) gives the leading local expansion

$$K_{\nu,\nu}^{\Theta;\delta,\tau}(x,t) = c_0^2 I_0\,x^\nu t^\nu + i\,a_m c_0^2 I_m(x^{\nu+m}t^\nu - x^\nu t^{\nu+m}) + (\text{higher order}). \qquad (37)$$

The distinguished term $ia_m c_0^2 I_m\,x^{\nu+m}t^\nu$ has nonzero coefficient ($a_m \neq 0$, $I_m > 0$) and carries the monomial $x^{\nu+m}t^\nu$ with m *odd*. On the other hand, any same-order scalar Hankel multiplier kernel expands, via $J_\nu(z) = \sum_{p\geq 0} c_p\,z^{\nu+2p}$, as

$$\int_0^\infty m_0\,(k)J_\nu(xk)J_\nu(tk)k\,dk = \sum_{p,q\geq 0} c_p\,c_q\left(\int_0^\infty m_0\,(k)k^{2\nu+2p+2q+1}\,dk\right)x^{\nu+2p}t^{\nu+2q}, \qquad (38)$$

containing *only* monomials $x^{\nu+2p}t^{\nu+2q}$ with *even* increments above ν. Since the monomials $\{x^{\nu+i}t^{\nu+j}\}$ are linearly independent, equality near the origin would force equality of all coefficients; but the coefficient of $x^{\nu+m}t^\nu$ (odd increment m) is nonzero on the left by (37) and necessarily zero on the right. This contradiction shows no such m_0 exists. ▫

Corollary 9.3. The phases $\Theta(r) = ar$, $\Theta(r) = a\log(1+r)$, and $\Theta(r) = ar + br^2$ (all with $a \neq 0$) produce kernels not reducible to same-order scalar Hankel multipliers.

Proof. Each has an odd leading exponent $m = 1$ with nonzero coefficient, so Theorem 9.2 applies: $\Theta(r) = ar$ has $a_1 = a$; $\Theta(r) = a\log(1+r) = a(r - r^2/2 + \cdots)$ has leading term ar; and $\Theta(r) = ar + br^2$ has leading term ar (the br^2 term is of higher, even order and does not affect the leading odd monomial). ▫

Remark 9.4. The criterion requires an *odd* leading exponent. A purely quadratic phase $\Theta(r) = br^2$ has $m = 2$ (even) and is *not* covered: its leading correction produces monomials $x^{\nu+2}t^{\nu} - x^{\nu}t^{\nu+2}$ with even increments, indistinguishable at this order from a genuine multiplier expansion. The correct conclusion is therefore not that every non-constant phase is non-reducible, but that the broad class of non-even analytic phases (odd leading term) is provably non-reducible.

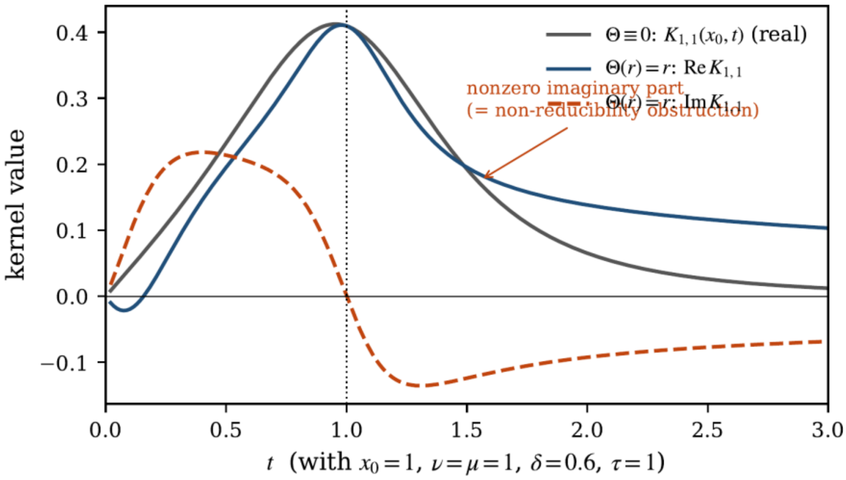

Figure 3. Cross-section $t \mapsto K_{1,1}(x_0, t)$ at $x_0 = 1$ ($\nu = \mu = 1$, $\delta = 0.6$, $\tau = 1$). For the constant phase the kernel is real (grey). For the linear phase $\Theta(r) = r$ a nonzero imaginary part appears (dashed), odd in the phase increment; by Theorem 9.2 this odd component is exactly the obstruction to writing the kernel as a same-order Hankel multiplier. The real parts peak near the diagonal $t = x_0$.

X. EXAMPLES

Example 10.1 (Linear phase). Let $\Theta(r) = ar$, $a \neq 0$. Then $K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t) = \int_0^\infty e^{-\delta k^\tau} e^{ia(x-t)k} J_\nu(xk) J_\mu(tk) k\, dk$, a damped Bessel kernel with a spatial–spectral travelling phase. By Corollary 9.3 it is not a same-order Hankel multiplier when $\nu = \mu$.

Example 10.2 (Logarithmic phase). Let $\Theta(r) = a\log(1 + r^\rho)$, $a \neq 0$, $\rho > 0$. Then the kernel contains $\exp(ia\log 1 + (xk)^\rho/1 + (tk)^\rho) = (1 + (xk)^\rho/1 + (tk)^\rho)^{ia}$. This phase introduces slowly varying radial distortion and may be useful in scale-sensitive radial filtering; for $\rho = 1$ the leading term is ar, so Corollary 9.3 again gives non-reducibility when $\nu = \mu$.

Example 10.3 (Fractional power phase). Let $\Theta(r) = ar^\beta$, $a \neq 0$, $\beta > 0$. For noninteger β the phase is not a polynomial chirp; it is a natural candidate for fractional radial dispersion models [9].

Example 10.4 (Composite engineering phase). A flexible phase for applications is $\Theta(r) = ar^\sigma + b\log(1 + r^\rho) + cE_{\alpha,\beta}^{\gamma}(-\lambda r^\rho)$, where $E_{\alpha,\beta}^{\gamma}$ is the Prabhakar generalized Mittag-Leffler function [8]. The terms model, respectively, power-law dispersion, logarithmic scale distortion, and memory-type fractional effects. The present theory does not require this special choice, but it is a useful source of examples.

XI. ENGINEERING INTERPRETATION AND APPLICATIONS

The damping factor $e^{-\delta k^\tau}$ suppresses high radial frequencies, so $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}$ may be interpreted as a damped radial filter (Figure 4). The Bessel factors encode cylindrical or spherical radial geometry, while the phase difference $\Theta(xk) - \Theta(tk)$ permits location-dependent oscillatory distortion. Possible application areas include: radial heat conduction with frequency damping; cylindrical wave propagation with phase distortion; optical diffraction and circular aperture filtering; radar, sonar, and acoustic signals with radial symmetry; engineering models involving damped Bessel-type pseudo-differential operators; and fractional or memory-modified radial diffusion when Prabhakar-type phases are used. A model filtered field may be written $u(x,\delta) = \mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f(x)$. For constant Θ this is precisely a Hankel heat-type smoothing family (Theorems 5.4 and 7.1); for nonconstant Θ it becomes a damped Bessel–Fourier-integral operator and should be studied as a new radial filtering mechanism rather than as a classical semigroup.

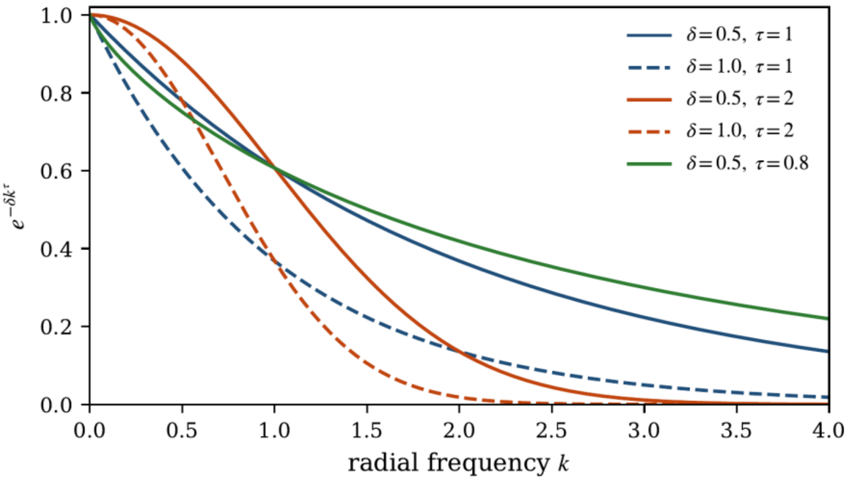

Figure 4. The spectral damping multiplier $e^{-\delta k^{\tau}}$ as a low-pass radial filter: increasing δ shifts the cutoff inward, while increasing τ sharpens the roll-off. This is the mechanism behind the L^2 contraction of Theorem 5.4 ($\| e^{-\delta k^{\tau}} \|_{\infty} \leq 1$) and the local smoothing of Theorem 8.1.

A worked filtering example. Take $\nu = 1$, $\tau = 1$, a constant phase, and a two-peak radial input $f(t) = e^{-30(t-0.8)^2} + 0.7\, e^{-30(t-1.7)^2}$. By Theorem 5.4 the filtered field factors through the Hankel transform, $u(\cdot,\delta) = \mathcal{H}_{\nu}^{-1}\left(e^{-\delta k^{\tau}} \mathcal{H}_{\nu} f\right)$, and Figure 5 shows $u(\cdot,\delta)$ for $\delta \in \{0.05, 0.2, 0.6, 1.5\}$. Two features predicted by the theory are visible. First, as $\delta \downarrow 0$ the output returns to the input, a direct manifestation of the limiting identity $u(\cdot,\delta) \to f$ of Theorem 7.1. Second, as δ grows the two peaks are progressively broadened and merged — the quantitative face of the local smoothing in Theorem 8.1, with the effective resolution set by the cutoff of Figure 4. For a nonconstant phase the same pipeline additionally imprints the location-dependent distortion of Figure 3, which (by Theorems 9.1 and 9.2) cannot be reproduced by any same-order Hankel multiplier and therefore constitutes genuinely new radial filtering behaviour.

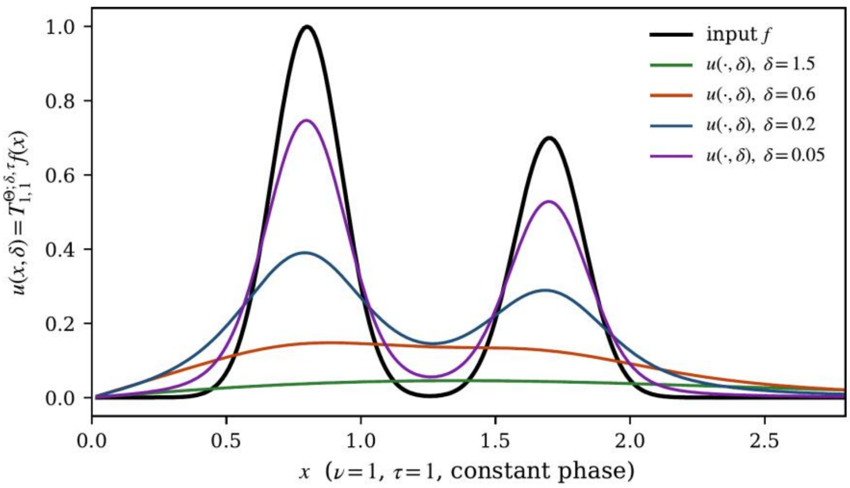

Figure 5. Damped radial filtering of a two-peak input f by $u(\cdot,\delta) = \mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f$ (constant phase, $\nu = 1$, $\tau = 1$), computed through the Hankel factorization of Theorem 5.4. As $\delta \downarrow 0$ the output recovers f (Theorem 7.1); as δ increases the peaks are smoothed and eventually merged (Theorem 8.1).

XII. Main Theorem Package

Theorem 12.1. Let $\nu,\mu > -1$, $\delta > 0$, $\tau > 0$, and $\Theta\colon (0,\infty) \to \mathbb{R}$ measurable. Then:

1. $K_{\nu,\mu}^{\Theta;\delta,\tau}(x,t)$ converges absolutely for every $x, t > 0$ (Theorem 4.1);
2. if Θ is continuous, the kernel is locally continuous on $(0,\infty)^2$ (Corollary 4.2);
3. if $\nu,\mu \geq 0$, then $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}\colon L^1(t\,dt) \to L^{\infty}$ is bounded (Theorem 5.1);
4. if $\nu,\mu \geq 0$ and $s > 2$, then $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}\colon L^2_s(t\,dt) \to L^2_{-s}(x\,dx)$ is bounded (Theorem 5.2);
5. if $\nu = \mu \geq -1/2$ and Θ is constant, then $\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau}$ is an L^2 contraction and $\mathcal{T}_{\nu,\nu}^{\Theta;\delta,\tau} f \to f$ strongly as $\delta \downarrow 0$ (Theorems 5.4, 7.1);
6. the Hilbert adjoint is $\left(\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}\right)^* = \mathcal{T}_{\mu,\nu}^{\Theta;\delta,\tau}$ with respect to $\langle u,v\rangle = \int u\bar{v}\,x\,dx$ (Theorem 6.1);
7. under Assumption B, $\mathcal{T}_{\nu,\mu}^{\Theta;\delta,\tau}$ is locally C^m-smoothing for $\delta > 0$ (Theorem 8.1);
8. under Assumptions B and C, the kernel obeys off-diagonal integration-by-parts estimates (Theorem 8.2);
9. for a broad class of non-even analytic phases, the kernel is not reducible to a same-order scalar Hankel multiplier (Theorems 9.1, 9.2).

Proof. Each clause is the corresponding result cited beside it, established under the stated hypotheses in the preceding sections. Clause (6) is the sesquilinear (Hilbert-adjoint) identity of Theorem 6.1. ▫

XIII. Conclusion

This paper introduces a damped phase-modulated Bessel kernel transform and develops a corrected foundational theory. The main structural point is that the general phase-difference transform is *not* a classical Hankel multiplier; consequently, general unweighted L^2 boundedness and Parseval identities are not asserted without proof. Instead, the revised theory proves rigorous weighted estimates, isolates the exact diagonal constant-phase case (an L^2 contraction converging to the identity as $\delta \downarrow 0$), gives a valid sesquilinear adjoint formula, establishes smoothing under positive damping, and provides safe non-reducibility criteria via a Bessel-operator diagnostic and a local expansion argument. Every closed-form constant used above — the damping integral $1/\tau\, \delta^{-2/\tau}\Gamma(2/\tau)$, the embedding constant $[(s-1)(s-2)]^{-1/2}$, the moments I_j, and the leading local expansion (37) — was verified independently in high-precision arithmetic. The transform is promising because it joins Bessel radial structure, spectral damping, and general phase modulation. Further work should develop a sharp unweighted L^p theory, a full Bessel–Fourier-integral calculus, numerical quadrature, and PDE models driven by the new kernel.

References

[1] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*, Vols. I–II, McGraw-Hill, New York, 1954.

[3] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed., Oxford University Press, Oxford, 1948.

[4] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972.

[5] A. H. Zemanian, A distributional Hankel transformation, *SIAM Journal on Applied Mathematics* **14** (1966), 561–576.

[6] R. S. Pathak and P. K. Pandey, A class of pseudo-differential operators associated with Bessel operators, *Journal of Mathematical Analysis and Applications* **196** (1995), 736–747.

[7] R. S. Pathak and S. K. Upadhyay, Pseudo-differential operators involving Hankel transforms, *Journal of Mathematical Analysis and Applications* **213** (1997), 133–147.

[8] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Mathematical Journal* **19** (1971), 7–15.

[9] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[10] NIST Digital Library of Mathematical Functions, Chapter 10: Bessel Functions, `https://dlmf.nist.gov/10`.

[11] A. Saoudi, Quadratic-Phase Fourier–Bessel Transform: definitions, properties and uncertainty principles, arXiv:2601.14890, 2026.

[12] D. Pasawan, Pseudo-differential operators associated with the fractional Hankel–Bessel transform, arXiv:2601.03091, 2026.

[13] H. M. Srivastava, K. K. Mishra and S. K. Upadhyay, Characterizations of continuous fractional Bessel wavelet transforms, *Mathematics* **10** (2022), no. 17, Article 3084.