



PYTHAGOREAN m-POLAR INTERVAL VALUED FUZZY NEUTROSOPHIC TOPOLOGY

Pavithra K¹, Nirmala Irudayam F²

PG student¹, Assistant Professor²

Department of mathematics

Nirmala college for women

Coimbatore

India

Abstract : This paper focuses on introducing the new idea of Pythagorean m-polar interval valued fuzzy neutrosophic set and its corresponding topology. This new set serves as a hybrid model combining the ideas of Pythagorean fuzzy sets, interval valued neutrosophic fuzzy sets and m-polar fuzzy sets. In this article we present some basic properties of the newly framed set and hence form the Pythagorean m-polar interval valued fuzzy neutrosophic topology

IndexTerms - : Pythagorean m-polar interval valued fuzzy neutrosophic set; Pythagorean m-polar interval valued fuzzy neutrosophic topology.

INTRODUCTION

In 1965, the thought of fuzzy sets as an augmentation of the conventional crisp set was inaugurated by Zadeh [18], to overcome these deficiencies, by associating the membership function $\mu_A : X \rightarrow [0,1]$. The notion of intuitionistic fuzzy set (IFS) was initiated by Atanassov[2,3] as a significant generalization of fuzzy set. Intuitionistic fuzzy sets are very useful in situations when description of a problem by a linguistic variable, given in terms of a membership function only, seems too complicated. In 2005, the model of neutrosophic sets, which is the broad view of intuitionistic fuzzy sets, for handling with difficulties involving exaggeration, indeterminacy and irregularity was explored by Smarandache [13]. The main aspiration behind this article is to study some features of Pythagorean m-polar fuzzy neutrosophic sets and construct topology on it. Wang et al. [16] proposed another extension of neutrosophic set which is single valued neutrosophic. Also Wang et al. [17] introduced the notion of interval valued neutrosophic set which is an instance of neutrosophic set. The rest of the paper is systemized as: Elementary notions are dealt with in Section 2. Section 3 presents some notions of Pythagorean m-polar fuzzy neutrosophic sets. The topological structure on our proposed model along with its prime attributes is presented in Section 4.

NEED OF THE STUDY.

1. Generalizing classical topology: Pythagorean m-polar interval-valued fuzzy neutrosophic topology provides a more comprehensive and flexible framework for topological spaces.
2. Modeling complex systems: This mathematical structure can be used to model complex systems with multiple attributes, uncertainties, and indeterminacies.
3. Applications in various fields: Potential applications in fields like artificial intelligence, data analysis, decision-making, and engineering.
4. Developing new mathematical tools: Studying this topology can lead to the development of new mathematical tools and techniques for solving problems in various domains.

RESEARCH METHODOLOGY

Conduct theoretical analysis including their properties and behavior, Develop computational methods and algorithms for working with Pythagorean m-polar interval-valued fuzzy neutrosophic topological spaces.

PRELIMINARIES

Definition 2.1. [18] A collection of orderly pairs $(h, T_F(h))$, h being an element of the underlying universe X and T_F (the affiliation, association or membership function) is a well defined map, that drives members of X to $[0,1]$, is entitled as a *fuzzy set* (FS) F over X . In other words

$$G = \{h, T(h), F(h) : h \in X\}$$
$$0 \leq T(\hbar) + F(\hbar) \leq 1.$$
$$P = \{ \langle h, T_n(h), F_n(h) \rangle : h \in X$$

The quantity $\varphi(h) = \sqrt{1 - \{T^2(h) + F^2(h)\}}$ is famous as the hesitation margin.

$$N = \{ \langle h, T_N(h), I_N(h), F_N(h) \rangle : h \in X \}$$

Definition 2.5. [1] A fuzzy neutrosophic set (fn-set) over X is delineated as

$$A = \{ \langle h, T_A(h), I_A(h), F_A(h) \rangle : h \in X \}$$

where $T, I, F: X \rightarrow [0, 1]$ in such away that $0 \leq T_A(h) + I_A(h) + F_A(h) \leq 3$.

$$0 \leq \left(T_{\mathcal{P}}^{(i)}(\hbar)\right)^2 + \left(F_{\mathcal{P}}^{(i)}(\hbar)\right)^2 \leq 1$$

For integral values of i ranging from 1 to m . A PmFS may be articulated as

$$\mathcal{P} = \left\{ \langle h, \left((T_{\mathcal{P}}^{(1)}(h), F_{\mathcal{P}}^{(1)}(h)), \dots, (T_{\mathcal{P}}^{(m)}(h), F_{\mathcal{P}}^{(m)}(h)) \right) \rangle : h \in X \right\}$$

Definition 2.7. Let $X \neq \emptyset$ be a crisp set. A family τ of subsets of X is called a *topology* on X if

- (i) φ and X itself belong to τ .
- (ii) The union of any number of members of τ is again in τ .
- (iii) The intersection of any finite number of members of τ belong to τ .

If τ is a topology on X , then (X, τ) is known as a *topological space*.

In this section, we introduce the notion of Pythagorean m-polar interval valued fuzzy neutrosophic set along with its prime characteristics and illustrations.

Definition 3.1. A Pythagorean m-polar interval valued fuzzy neutrosophic set (PmIVFNS) τ over a basic set X is marked by three mappings $T_{\tau}^i: X \rightarrow [0,1]^m$, $I_{\tau}^i: X \rightarrow [0,1]^m$ and $F_{\tau}^i: X \rightarrow [0,1]^m$, Where m is a natural number, $\forall i=1,2,\dots,m$ with the limitation that,

$$0 \leq \inf (T_\tau^i(h))^2 + \inf (I_\tau^i(h))^2 + \inf (F_\tau^i(h))^2 \leq 3$$

$$0 \leq \sup (T_\tau^i(h))^2 + \sup (I_\tau^i(h))^2 + \sup (F_\tau^i(h))^2 \leq 3$$

for all $h \in X$.

A PmIVFNS may be expressed as

$$\tau = \{(h, [\inf T_\tau^1(h), \sup T_\tau^1(h)], [\inf I_\tau^1(h), \sup I_\tau^1(h)], [\inf F_\tau^1(h), \sup F_\tau^1(h)], \dots, \dots, \dots)\}$$

$$[\inf T_\tau^m(h), \sup T_\tau^m(h)], [\inf I_\tau^m(h), \sup I_\tau^m(h)], [\inf F_\tau^m(h), \sup F_\tau^m(h)] : h \in X\}$$

If cardinality of X is l , then tabular structure of τ is as in Table 1:

TABLE 1. Tabular representation of PmIVFNS τ

τ				
h_1	$([infT_{\tau}^1(h_1), supT_{\tau}^1(h_1)], [infI_{\tau}^1(h_1), supI_{\tau}^1(h_1)], [infF_{\tau}^1(h_1), supF_{\tau}^1(h_1)])$	$([infT_{\tau}^2(h_1), supT_{\tau}^2(h_1)], [infI_{\tau}^2(h_1), supI_{\tau}^2(h_1)], [infF_{\tau}^2(h_1), supF_{\tau}^2(h_1)])$	...	$([infT_{\tau}^m(h_1), supT_{\tau}^m(h_1)], [infI_{\tau}^m(h_1), supI_{\tau}^m(h_1)], [infF_{\tau}^m(h_1), supF_{\tau}^m(h_1)])$
h_2	$([infT_{\tau}^1(h_2), supT_{\tau}^1(h_2)], [infI_{\tau}^1(h_2), supI_{\tau}^1(h_2)], [infF_{\tau}^1(h_2), supF_{\tau}^1(h_2)])$	$([infT_{\tau}^2(h_2), supT_{\tau}^2(h_2)], [infI_{\tau}^2(h_2), supI_{\tau}^2(h_2)], [infF_{\tau}^2(h_2), supF_{\tau}^2(h_2)])$	...	$([infT_{\tau}^m(h_2), supT_{\tau}^m(h_2)], [infI_{\tau}^m(h_2), supI_{\tau}^m(h_2)], [infF_{\tau}^m(h_2), supF_{\tau}^m(h_2)])$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
h_i	$([infT_{\tau}^1(h_i), supT_{\tau}^1(h_i)], [infI_{\tau}^1(h_i), supI_{\tau}^1(h_i)], [infF_{\tau}^1(h_i), supF_{\tau}^1(h_i)])$	$([infT_{\tau}^2(h_i), supT_{\tau}^2(h_i)], [infI_{\tau}^2(h_i), supI_{\tau}^2(h_i)], [infF_{\tau}^2(h_i), supF_{\tau}^2(h_i)])$	...	$([infT_{\tau}^m(h_i), supT_{\tau}^m(h_i)], [infI_{\tau}^m(h_i), supI_{\tau}^m(h_i)], [infF_{\tau}^m(h_i), supF_{\tau}^m(h_i)])$

The corresponding matrix format is

$$\tau = \begin{pmatrix} ([infT_{\tau}^1(h_1), supT_{\tau}^1(h_1)], [infI_{\tau}^1(h_1), supI_{\tau}^1(h_1)], [infF_{\tau}^1(h_1), supF_{\tau}^1(h_1)]) & ([infT_{\tau}^2(h_1), supT_{\tau}^2(h_1)], [infI_{\tau}^2(h_1), supI_{\tau}^2(h_1)], [infF_{\tau}^2(h_1), supF_{\tau}^2(h_1)]) & \dots & ([infT_{\tau}^m(h_1), supT_{\tau}^m(h_1)], [infI_{\tau}^m(h_1), supI_{\tau}^m(h_1)], [infF_{\tau}^m(h_1), supF_{\tau}^m(h_1)]) \\ ([infT_{\tau}^1(h_2), supT_{\tau}^1(h_2)], [infI_{\tau}^1(h_2), supI_{\tau}^1(h_2)], [infF_{\tau}^1(h_2), supF_{\tau}^1(h_2)]) & ([infT_{\tau}^2(h_2), supT_{\tau}^2(h_2)], [infI_{\tau}^2(h_2), supI_{\tau}^2(h_2)], [infF_{\tau}^2(h_2), supF_{\tau}^2(h_2)]) & \dots & ([infT_{\tau}^m(h_2), supT_{\tau}^m(h_2)], [infI_{\tau}^m(h_2), supI_{\tau}^m(h_2)], [infF_{\tau}^m(h_2), supF_{\tau}^m(h_2)]) \\ \vdots & \vdots & \ddots & \vdots \\ ([infT_{\tau}^1(h_i), supT_{\tau}^1(h_i)], [infI_{\tau}^1(h_i), supI_{\tau}^1(h_i)], [infF_{\tau}^1(h_i), supF_{\tau}^1(h_i)]) & ([infT_{\tau}^2(h_i), supT_{\tau}^2(h_i)], [infI_{\tau}^2(h_i), supI_{\tau}^2(h_i)], [infF_{\tau}^2(h_i), supF_{\tau}^2(h_i)]) & \dots & ([infT_{\tau}^m(h_i), supT_{\tau}^m(h_i)], [infI_{\tau}^m(h_i), supI_{\tau}^m(h_i)], [infF_{\tau}^m(h_i), supF_{\tau}^m(h_i)]) \end{pmatrix}$$

This $l \times m$ matrix is known as PmIVFN matrix. The assortment of each PmIVFNS characterized over universe would be designated by PmIVFNS(X).

Example 3.2. If $X=\{e,f\}$ be a crisp set, then

$$\tau = \begin{pmatrix} \begin{matrix} e \\ ([0.2,0.4], [0.6,0.3], [0.3,0.5]), ([0.6,0.8], [0.5,0.6], [0.1,0.4]), ([0.3,0.7], [0.7,0.5], [0.8,0.3]) \end{matrix} \\ \begin{matrix} f \\ ([0.1,0.9], [0.3,0.4], [0.5,0.7]), ([0.4,0.6], [0.9,0.3], [0.5,0.5]), ([0.6,0.2], [0.7,0.8], [0.8,0.5]) \end{matrix} \end{pmatrix}$$

TABLE 2. Tabular representation of P3IVFNS τ

τ			
e	$([0.2,0.4], [0.6,0.3], [0.3,0.5])$	$([0.6,0.8], [0.5,0.6], [0.1,0.4])$	$([0.3,0.7], [0.7,0.5], [0.8,0.3])$
f	$([0.1,0.9], [0.3,0.4], [0.5,0.7])$	$([0.4,0.6], [0.9,0.3], [0.5,0.5])$	$([0.6,0.2], [0.7,0.8], [0.8,0.5])$

The matrix form of this set is

$$\tau = \begin{pmatrix} ([0.2,0.4], [0.6,0.3], [0.3,0.5]) & ([0.6,0.8], [0.5,0.6], [0.1,0.4]) & ([0.3,0.7], [0.7,0.5], [0.8,0.3]) \\ ([0.1,0.9], [0.3,0.4], [0.5,0.7]) & ([0.4,0.6], [0.9,0.3], [0.5,0.5]) & ([0.6,0.2], [0.7,0.8], [0.8,0.5]) \end{pmatrix}$$

Definition 3.3 . Let τ_1 and τ_2 be PmIVFNS over X. τ_1 is acknowledged as a *subset* of τ_2 , written as $\tau_1 \subseteq \tau_2, \forall \tau \in X$ and each values of i ranging from 1 to m , if

$$\begin{aligned} infT_{\tau_1}^i(h) &\leq infT_{\tau_2}^i(h), \quad supT_{\tau_1}^i(h) \leq supT_{\tau_2}^i(h) \\ infI_{\tau_1}^i(h) &\leq infI_{\tau_2}^i(h), \quad supI_{\tau_1}^i(h) \leq supI_{\tau_2}^i(h) \\ infF_{\tau_1}^i(h) &\leq infF_{\tau_2}^i(h), \quad supF_{\tau_1}^i(h) \leq supF_{\tau_2}^i(h) \end{aligned}$$

τ_1 & τ_2 are said to be equal if $\tau_1 \subseteq \tau_2 \subseteq \tau_1$ and is written as $\tau_1 = \tau_2$

Example 3.4. Let

$$\begin{aligned} \tau_1 &= \begin{pmatrix} ([0.4,0.5], [0.3,0.8], [0.7,0.5]) & ([0.3,0.6], [0.1,0.5], [0.2,0.8]) & ([0.9,0.7], [0.3,0.7], [0.3,0.6]) \\ ([0.7,0.1], [0.3,0.9], [0.6,0.4]) & ([0.2,0.6], [0.1,0.8], [0.5,0.6]) & ([0.3,0.5], [0.9,0.6], [0.3,0.9]) \\ ([0.2,0.5], [0.7,0.8], [0.8,0.8]) & ([0.4,0.6], [0.5,0.6], [0.8,0.9]) & ([0.5,0.5], [0.4,0.9], [0.2,0.9]) \end{pmatrix} \text{ and} \\ \tau_2 &= \begin{pmatrix} ([0.6,0.8], [0.2,0.4], [0.5,0.3]) & ([0.7,0.9], [0.0,0.2], [0.1,0.6]) & ([1.0,0.8], [0.2,0.6], [0.1,0.4]) \\ ([0.9,0.5], [0.2,0.5], [0.4,0.1]) & ([0.9,0.8], [0.0,0.6], [0.4,0.3]) & ([0.8,1.0], [0.5,0.4], [0.1,0.7]) \\ ([1.0,0.8], [0.6,0.4], [0.5,0.1]) & ([0.9,0.7], [0.3,0.5], [0.6,0.7]) & ([0.9,0.6], [0.3,0.4], [0.1,0.7]) \end{pmatrix} \end{aligned}$$

be PmIVFNS over some set x, then $\tau_1 \subseteq \tau_2$

Definition 3.5. A PmIVFNS τ over X is known as *null* PmIVFNS if $\inf T_{\tau}^i(h) = 0, \sup T_{\tau}^i(h) = 0; \inf I_{\tau}^i(h) = 1, \sup I_{\tau}^i(h) = 1$ and $\inf F_{\tau}^i(h) = 1, \sup F_{\tau}^i(h) = 1, \forall h \in X$ and all acceptable values of i . It is designated by ϕ .

$$\text{Thus, } \phi = \begin{pmatrix} ([0,0], [1,1], [1,1]) & ([0,0], [1,1], [1,1]) & \dots & ([0,0], [1,1], [1,1]) \\ ([0,0], [1,1], [1,1]) & ([0,0], [1,1], [1,1]) & \dots & ([0,0], [1,1], [1,1]) \\ \vdots & \vdots & \ddots & \vdots \\ ([0,0], [1,1], [1,1]) & ([0,0], [1,1], [1,1]) & \dots & ([0,0], [1,1], [1,1]) \end{pmatrix}$$

Definition 3.6. A PmIVFNS τ over X is known as *absolute* PmIVFNS if $\inf T_{\tau}^i(h) = 1, \sup T_{\tau}^i(h) = 1; \inf I_{\tau}^i(h) = 0, \sup I_{\tau}^i(h) = 0$ and $\inf F_{\tau}^i(h) = 0, \sup F_{\tau}^i(h) = 0, \forall h \in X$. It is denoted by κ .

$$\kappa = \begin{pmatrix} ([1,1], [0,0], [0,0]) & ([1,1], [0,0], [0,0]) & \dots & ([1,1], [0,0], [0,0]) \\ ([1,1], [0,0], [0,0]) & ([1,1], [0,0], [0,0]) & \dots & ([1,1], [0,0], [0,0]) \\ \vdots & \vdots & \ddots & \vdots \\ ([1,1], [0,0], [0,0]) & ([1,1], [0,0], [0,0]) & \dots & ([1,1], [0,0], [0,0]) \end{pmatrix}$$

Definition 3.7. The *complement* of a PmIVFNS

$$\tau = \left\{ \frac{h}{([\inf T_{\tau}^i(h), \sup T_{\tau}^i(h)], [\inf I_{\tau}^i(h), \sup I_{\tau}^i(h)], [\inf F_{\tau}^i(h), \sup F_{\tau}^i(h)])} : h \in X, i = 1, \dots, m \right\}$$

over X is defined as

$$\tau^c = \left\{ \frac{h}{([\inf F_{\tau}^i(h), \sup F_{\tau}^i(h)], 1 - [\inf I_{\tau}^i(h), \sup I_{\tau}^i(h)], [\inf T_{\tau}^i(h), \sup T_{\tau}^i(h)])} : h \in X, i = 1, \dots, m \right\}.$$

Example 3.8. The complement of the PmIVFNS τ given in example 3.2 is

$$\tau^c = \begin{pmatrix} ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4]) & ([0.1, 0.4], [0.5, 0.4], [0.6, 0.8]) & ([0.8, 0.3], [0.3, 0.5], [0.3, 0.7]) \\ ([0.5, 0.7], [0.7, 0.6], [0.1, 0.9]) & ([0.5, 0.5], [0.1, 0.7], [0.4, 0.6]) & ([0.8, 0.5], [0.3, 0.2], [0.6, 0.2]) \end{pmatrix}$$

Definition 3.9. The union of any PmIVFNSs τ_1 and τ_2 expressed over the same universe X is represented as

$$\tau_1 \cup_m \tau_2 = \left\{ \frac{h}{([\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)), \max(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h))], [\min(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)), \min(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h))], [\min(\inf F_{\tau_1}^i(h), \inf F_{\tau_2}^i(h)), \min(\sup F_{\tau_1}^i(h), \sup F_{\tau_2}^i(h))])} : h \in X, i = 1, \dots, m \right\},$$

Definition 3.10. The intersection of any PmIVFNSs τ_1 and τ_2 expressed over the same universe X is represented as

$$\tau_1 \cap_m \tau_2 = \left\{ \frac{h}{([\min(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)), \min(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h))], [\max(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)), \max(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h))], [\max(\inf F_{\tau_1}^i(h), \inf F_{\tau_2}^i(h)), \max(\sup F_{\tau_1}^i(h), \sup F_{\tau_2}^i(h))])} : h \in X, i = 1, \dots, m \right\}$$

Example 3.11. If

$$\tau_1 = \begin{pmatrix} ([0.5, 0.2], [0.2, 0.3], [0.7, 0.8]) & ([0.1, 0.5], [0.9, 1.0], [0.8, 0.2]) & ([0.3, 0.6], [0.4, 0.2], [0.1, 0.3]) \\ ([0.6, 0.1], [0.0, 0.3], [0.4, 0.7]) & ([0.9, 0.1], [0.3, 0.7], [0.5, 1.0]) & ([0.2, 0.3], [0.8, 0.6], [0.2, 0.9]) \\ ([0.8, 0.2], [0.6, 0.3], [0.5, 0.1]) & ([0.6, 0.6], [0.2, 0.1], [1.0, 0.7]) & ([0.5, 0.2], [0.9, 0.7], [0.3, 0.8]) \end{pmatrix}$$

and

$$\tau_2 = \begin{pmatrix} ([0.4, 0.7], [0.9, 0.2], [0.8, 0.0]) & ([0.5, 0.8], [0.2, 0.6], [0.7, 0.7]) & ([0.0, 0.6], [0.5, 0.9], [0.9, 0.8]) \\ ([0.6, 0.3], [0.1, 0.9], [0.8, 0.2]) & ([0.7, 0.2], [0.4, 0.9], [0.9, 0.2]) & ([0.4, 0.8], [0.6, 0.4], [0.9, 1.0]) \\ ([0.4, 0.5], [0.2, 0.6], [0.5, 0.5]) & ([0.5, 0.2], [0.1, 0.8], [0.4, 0.4]) & ([0.6, 0.3], [0.8, 0.2], [0.9, 0.4]) \end{pmatrix}$$

are two PmIVFNS defined over the same universe of discourse X , then

$$\tau_1 \cup_m \tau_2 = \begin{pmatrix} ([0.5, 0.7], [0.2, 0.2], [0.7, 0.0]) & ([0.5, 0.8], [0.2, 0.6], [0.7, 0.2]) & ([0.0, 0.6], [0.4, 0.2], [0.1, 0.3]) \\ ([0.6, 0.3], [0.0, 0.3], [0.4, 0.2]) & ([0.9, 0.2], [0.3, 0.7], [0.5, 0.2]) & ([0.4, 0.8], [0.6, 0.4], [0.2, 0.9]) \\ ([0.8, 0.5], [0.2, 0.3], [0.5, 0.1]) & ([0.6, 0.6], [0.1, 0.1], [0.4, 0.4]) & ([0.6, 0.3], [0.9, 0.7], [0.9, 0.8]) \end{pmatrix} \text{ and}$$

$$\tau_1 \cap_m \tau_2 = \begin{pmatrix} ([0.4, 0.2], [0.9, 0.3], [0.8, 0.8]) & ([0.1, 0.5], [0.9, 1.0], [0.8, 0.7]) & ([0.0, 0.6], [0.5, 0.9], [0.9, 0.8]) \\ ([0.6, 0.1], [0.1, 0.9], [0.8, 0.7]) & ([0.7, 0.1], [0.4, 0.9], [0.9, 1.0]) & ([0.2, 0.3], [0.8, 0.6], [0.9, 1.0]) \\ ([0.4, 0.2], [0.6, 0.6], [0.5, 0.5]) & ([0.5, 0.2], [0.2, 0.8], [1.0, 0.7]) & ([0.5, 0.2], [0.9, 0.7], [0.9, 0.8]) \end{pmatrix}$$

Proposition 3.12. If $\tau, \tau_1, \tau_2, \tau_3$ are PmIVFNS over X , then

1. $\varphi \cup_m \tau = \tau$
2. $\varphi \cap_m \tau = \varphi$
3. $\kappa \cup_m \tau = \kappa$
4. $\kappa \cap_m \tau = \kappa$
5. $\tau \cup_m \tau = \tau$
6. $\tau \cap_m \tau = \tau$
7. $\tau_1 \cup_m \tau_2 = \tau_2 \cup_m \tau_1$
8. $\tau_1 \cap_m \tau_2 = \tau_2 \cap_m \tau_1$
9. $\tau_1 \cup_m (\tau_2 \cup_m \tau_3) = (\tau_1 \cup_m \tau_2) \cup_m \tau_3$
10. $\tau_1 \cap_m (\tau_2 \cap_m \tau_3) = (\tau_1 \cap_m \tau_2) \cap_m \tau_3$
11. $\tau_1 \cup_m (\tau_2 \cap_m \tau_3) = (\tau_1 \cup_m \tau_2) \cap_m (\tau_1 \cup_m \tau_3)$
12. $\tau_1 \cap_m (\tau_2 \cup_m \tau_3) = (\tau_1 \cap_m \tau_2) \cup_m (\tau_1 \cap_m \tau_3)$

Proof. Here, we prove only (11). We may assume without losing generality, that $\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)) = \inf T_{\tau_1}^i(h)$, $\max(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h)) = \sup T_{\tau_1}^i(h)$; $\max(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)) = \inf I_{\tau_1}^i(h)$, $\max(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h)) = \sup I_{\tau_1}^i(h)$; $\max(\inf F_{\tau_1}^i(h), \inf F_{\tau_2}^i(h)) = \inf F_{\tau_1}^i(h)$, $\max(\sup F_{\tau_1}^i(h), \sup F_{\tau_2}^i(h)) = \sup F_{\tau_1}^i(h)$.
Then, $\forall h \in X$ and $i=1, \dots, m$

$$\begin{aligned}
 & \left\{ \frac{h}{(\min(\inf T_{\tau_2}^i(h), \inf T_{\tau_3}^i(h)), \min(\sup T_{\tau_2}^i(h), \sup T_{\tau_3}^i(h)), [\max(\inf I_{\tau_2}^i(h), \inf I_{\tau_3}^i(h)), \max(\sup I_{\tau_2}^i(h), \sup I_{\tau_3}^i(h)), [\max(\inf F_{\tau_2}^i(h), \inf F_{\tau_3}^i(h)), \max(\sup F_{\tau_2}^i(h), \sup F_{\tau_3}^i(h))])]} \right\} \tau_2 \cup_m \tau_3 = \\
 & = \left\{ \frac{h}{([\inf T_{\tau_2}^i(h), \sup T_{\tau_2}^i(h)], [\inf I_{\tau_2}^i(h), \sup I_{\tau_2}^i(h)], [\inf F_{\tau_2}^i(h), \sup F_{\tau_2}^i(h)])} \right\} \\
 & \therefore \tau_1 \cup_m (\tau_2 \cap_m \tau_3) = \left\{ \frac{h}{([\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)], [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)], [\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)])} \right\} \cup_m \\
 & \left\{ \frac{h}{([\inf T_{\tau_2}^i(h), \sup T_{\tau_2}^i(h)], [\inf I_{\tau_2}^i(h), \sup I_{\tau_2}^i(h)], [\inf F_{\tau_2}^i(h), \sup F_{\tau_2}^i(h)])} \right\} \\
 & = \left\{ \frac{h}{([\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)), \max(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h)), [\min(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)), \min(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h)), [\min(\inf F_{\tau_1}^i(h), \inf F_{\tau_2}^i(h)), \min(\sup F_{\tau_1}^i(h), \sup F_{\tau_2}^i(h))])]} \right\} \\
 & = \left\{ \frac{h}{([\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)], [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)], [\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)])} \right\} \text{ and } \\
 & \tau_1 \cup_m \tau_2 = \left\{ \frac{h}{([\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)), \max(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h)), [\min(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)), \min(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h)), [\min(\inf F_{\tau_1}^i(h), \inf F_{\tau_2}^i(h)), \min(\sup F_{\tau_1}^i(h), \sup F_{\tau_2}^i(h))])]} \right\} \\
 & = \left\{ \frac{h}{([\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)], [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)], [\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)])} \right\} \\
 & \tau_1 \cup_m \tau_3 = \left\{ \frac{h}{([\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_3}^i(h)), \max(\sup T_{\tau_1}^i(h), \sup T_{\tau_3}^i(h)), [\min(\inf I_{\tau_1}^i(h), \inf I_{\tau_3}^i(h)), \min(\sup I_{\tau_1}^i(h), \sup I_{\tau_3}^i(h)), [\min(\inf F_{\tau_1}^i(h), \inf F_{\tau_3}^i(h)), \min(\sup F_{\tau_1}^i(h), \sup F_{\tau_3}^i(h))])]} \right\} \\
 & = \left\{ \frac{h}{([\inf T_{\tau_3}^i(h), \sup T_{\tau_3}^i(h)], [\inf I_{\tau_3}^i(h), \sup I_{\tau_3}^i(h)], [\inf F_{\tau_3}^i(h), \sup F_{\tau_3}^i(h)])} \right\}
 \end{aligned}$$

$$\begin{aligned}
(\tau_1 \cup_m \tau_2) \cap_m (\tau_1 \cup_m \tau_3) &= \left\{ \frac{h}{([\min(\inf T_{\tau_1}^i(h), \inf T_{\tau_3}^i(h)), \min(\sup T_{\tau_1}^i(h), \sup T_{\tau_3}^i(h))], [\max(\inf I_{\tau_1}^i(h), \inf I_{\tau_3}^i(h)), \max(\sup I_{\tau_1}^i(h), \sup I_{\tau_3}^i(h))])} \right\} \\
&= \left\{ \frac{h}{([\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)], [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)], [\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)])} \right\}
\end{aligned}$$

Corollary 3.13. (1) $\varphi \cup_m \kappa = \kappa$ (2) $\varphi \cap_m \kappa = \varphi$

Proposition 3.14. If τ_1 and τ_2 are $PmIVFNS_S$ over X , then

1. $\tau_1 \cap_m \tau_2 \subseteq \tau_1 \subseteq \tau_1 \cup_m \tau_2$
2. $\tau_1 \cap_m \tau_2 \subseteq \tau_2 \subseteq \tau_1 \cup_m \tau_2$

Proof. The results are easy consequences of properties of max and min.

Proposition 3.15. Let τ_1, τ_2 be $PmIVFNS_S$ over universe set X , then De Morgan laws hold i.e.

1. $(\tau_1 \cup_m \tau_2)^c = \tau_1^c \cap_m \tau_2^c$
2. $(\tau_1 \cap_m \tau_2)^c = \tau_1^c \cup_m \tau_2^c$

Proof. Here, we demonstrate only (1). The verification of (2) perhaps provided in the same way. We may assume, without losing the generality, that $\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)) = \inf T_{\tau_1}^i(h)$, $\max(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h)) = \sup T_{\tau_1}^i(h)$; $\max(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)) = \inf I_{\tau_1}^i(h)$, $\max(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h)) = \sup I_{\tau_1}^i(h)$ and $\max(\inf F_{\tau_1}^i(h), \inf F_{\tau_2}^i(h)) = \inf F_{\tau_1}^i(h)$, $\max(\sup F_{\tau_1}^i(h), \sup F_{\tau_2}^i(h)) = \sup F_{\tau_1}^i(h)$.

Then $\forall h \in X$ and $i=1, 2, \dots, m$.

$$\begin{aligned}
(\tau_1 \cup_m \tau_2)^c &= \left\{ \frac{h}{([\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)), \max(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h))], [\min(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)), \min(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h))])} \right\} \\
&= \left\{ \frac{h}{([\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)], [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)], [\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)])} \right\} \\
&= \left\{ \frac{h}{([\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)], \{1 - [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)]\}, [\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)])} \right\} \\
\tau_1^c \cap_m \tau_2^c &= \left\{ \frac{h}{([\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)], [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)], [\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)])} \right\}^c \cap_m \left\{ \frac{h}{([\inf T_{\tau_2}^i(h), \sup T_{\tau_2}^i(h)], [\inf I_{\tau_2}^i(h), \sup I_{\tau_2}^i(h)], [\inf F_{\tau_2}^i(h), \sup F_{\tau_2}^i(h)])} \right\} \\
&= \left\{ \frac{h}{([\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)], \{1 - [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)]\}, [\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)])} \right\}^c \cap_m \left\{ \frac{h}{([\inf F_{\tau_2}^i(h), \sup F_{\tau_2}^i(h)], \{1 - [\inf I_{\tau_2}^i(h), \sup I_{\tau_2}^i(h)]\}, [\inf T_{\tau_2}^i(h), \sup T_{\tau_2}^i(h)])} \right\} \\
&= \left\{ \frac{h}{([\min(\inf F_{\tau_1}^i(h), \inf F_{\tau_2}^i(h)), \min(\sup F_{\tau_1}^i(h), \sup F_{\tau_2}^i(h))], \{1 - [\min(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h)), \min(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h))]\}, [\max(\inf T_{\tau_1}^i(h), \inf T_{\tau_2}^i(h)), \max(\sup T_{\tau_1}^i(h), \sup T_{\tau_2}^i(h))])} \right\} \\
&= \left\{ \frac{h}{([\inf F_{\tau_1}^i(h), \sup F_{\tau_1}^i(h)], \{1 - [\inf I_{\tau_1}^i(h), \sup I_{\tau_1}^i(h)]\}, [\inf T_{\tau_1}^i(h), \sup T_{\tau_1}^i(h)])} \right\}
\end{aligned}$$

Remark 3.16. Let τ is a *PmFNS* over universe set X . Then

1. $\tau \cup_m \tau^c \neq \kappa$
2. $\tau \cap_m \tau^c \neq \varphi$

Proposition 3.17.

1. $\varphi^c = \kappa$
2. $\kappa^c = \varphi$

Proof. Straight forward.

Definition 3.18. The difference of two *PmIVFNS* τ_1 and τ_2 expressed over the same universe X is represented as

$$\tau_1 \setminus \tau_2 = \left\{ \begin{array}{c} h \\ \left[\begin{array}{c} \min \left(\inf T_{\tau_1}^i(h), \inf F_{\tau_2}^i(h) \right), \min \left(\sup T_{\tau_1}^i(h), \sup F_{\tau_2}^i(h) \right) \\ \min \left(\inf I_{\tau_1}^i(h), \inf I_{\tau_2}^i(h) \right), \\ \min \left(\sup I_{\tau_1}^i(h), \sup I_{\tau_2}^i(h) \right), \left[\max \left(\inf F_{\tau_1}^i(h), \inf T_{\tau_2}^i(h) \right), \max \left(\sup F_{\tau_1}^i(h), \sup T_{\tau_2}^i(h) \right) \right] \end{array} \right] \end{array} \right\}$$

$h \in X, i = 1, 2, \dots, m.$

Example 3.19. For τ_1 and τ_2 given in Example 3.11, we have

$$\tau_1 \setminus \tau_2 = \left(\begin{array}{ccc} ([0.5, 0.0], [0.2, 0.2], [0.7, 0.8]) & ([0.1, 0.5], [0.2, 0.6], [0.8, 0.8]) & ([0.3, 0.6], [0.4, 0.2], [0.1, 0.6]) \\ ([0.6, 0.1], [0.0, 0.3], [0.6, 0.7]) & ([0.9, 0.1], [0.3, 0.7], [0.7, 1.0]) & ([0.2, 0.3], [0.6, 0.4], [0.4, 0.9]) \\ ([0.5, 0.2], [0.2, 0.3], [0.5, 0.5]) & ([0.4, 0.4], [0.1, 0.1], [1.0, 0.7]) & ([0.5, 0.2], [0.8, 0.2], [0.6, 0.8]) \end{array} \right)$$

Definition 3.20. The symmetric difference of two *PmIVFNS* τ_1 and τ_2 is set of elements which are either in τ_1 or in τ_2 but not in both i.e.

$$\tau_1 \Delta \tau_2 = (\tau_1 \setminus \tau_2) \cup_m (\tau_2 \setminus \tau_1)$$

Example 3.22. Let

$$\tau_1 = \left(\begin{array}{ccc} ([0.5, 0.2], [0.2, 0.3], [0.7, 0.8]) & ([0.1, 0.5], [0.9, 1.0], [0.8, 0.2]) & ([0.3, 0.6], [0.4, 0.2], [0.1, 0.3]) \\ ([0.6, 0.1], [0.0, 0.3], [0.4, 0.7]) & ([0.9, 0.1], [0.3, 0.7], [0.5, 1.0]) & ([0.2, 0.3], [0.8, 0.6], [0.2, 0.9]) \\ ([0.8, 0.2], [0.6, 0.3], [0.5, 0.1]) & ([0.6, 0.6], [0.2, 0.1], [1.0, 0.7]) & ([0.5, 0.2], [0.9, 0.7], [0.3, 0.8]) \end{array} \right)$$

and

$$\tau_2 = \left(\begin{array}{ccc} ([0.4, 0.7], [0.9, 0.2], [0.8, 0.0]) & ([0.5, 0.8], [0.2, 0.6], [0.7, 0.7]) & ([0.0, 0.6], [0.5, 0.9], [0.9, 0.8]) \\ ([0.6, 0.3], [0.1, 0.9], [0.8, 0.2]) & ([0.7, 0.2], [0.4, 0.9], [0.9, 0.2]) & ([0.4, 0.8], [0.6, 0.4], [0.9, 1.0]) \\ ([0.4, 0.5], [0.2, 0.6], [0.5, 0.5]) & ([0.5, 0.2], [0.1, 0.8], [0.4, 0.4]) & ([0.6, 0.3], [0.8, 0.2], [0.9, 0.4]) \end{array} \right)$$

So that

$$\tau_1 \setminus \tau_2 = \left(\begin{array}{ccc} ([0.5, 0.0], [0.2, 0.2], [0.7, 0.8]) & ([0.1, 0.5], [0.2, 0.6], [0.8, 0.8]) & ([0.3, 0.6], [0.4, 0.2], [0.1, 0.6]) \\ ([0.6, 0.1], [0.0, 0.3], [0.6, 0.7]) & ([0.9, 0.1], [0.3, 0.7], [0.7, 1.0]) & ([0.2, 0.3], [0.6, 0.4], [0.4, 0.9]) \\ ([0.5, 0.2], [0.2, 0.3], [0.5, 0.5]) & ([0.4, 0.4], [0.1, 0.1], [1.0, 0.7]) & ([0.5, 0.2], [0.8, 0.2], [0.6, 0.8]) \end{array} \right)$$

and

$$\tau_2 \setminus \tau_1 = \left(\begin{array}{ccc} ([0.4, 0.7], [0.2, 0.2], [0.8, 0.2]) & ([0.5, 0.2], [0.2, 0.6], [0.7, 0.7]) & ([0.0, 0.3], [0.4, 0.2], [0.9, 0.8]) \\ ([0.4, 0.3], [0.0, 0.3], [0.8, 0.2]) & ([0.5, 0.2], [0.3, 0.7], [0.9, 0.2]) & ([0.2, 0.8], [0.6, 0.4], [0.9, 1.0]) \\ ([0.4, 0.1], [0.2, 0.3], [0.8, 0.5]) & ([0.5, 0.2], [0.1, 0.1], [0.6, 0.6]) & ([0.3, 0.3], [0.8, 0.2], [0.9, 0.4]) \end{array} \right)$$

$$\begin{aligned} \therefore (\tau_1 \setminus \tau_2) \cup_m (\tau_2 \setminus \tau_1) &= \left(\begin{array}{ccc} ([0.5, 0.7], [0.2, 0.2], [0.7, 0.2]) & ([0.5, 0.5], [0.2, 0.6], [0.7, 0.7]) & ([0.3, 0.6], [0.4, 0.2], [0.1, 0.6]) \\ ([0.6, 0.3], [0.0, 0.3], [0.6, 0.2]) & ([0.9, 0.2], [0.3, 0.7], [0.7, 0.2]) & ([0.2, 0.8], [0.6, 0.4], [0.4, 0.9]) \\ ([0.5, 0.2], [0.2, 0.3], [0.5, 0.5]) & ([0.5, 0.4], [0.1, 0.1], [0.6, 0.6]) & ([0.5, 0.3], [0.8, 0.2], [0.6, 0.4]) \end{array} \right) \\ &= \tau_1 \Delta \tau_2 \end{aligned}$$

Definition 3.22. The sum of two *PmIVFNS* τ_1 and τ_2 chosen from same universe X is represented as

$$\tau_1 \oplus \tau_2 = \left\{ \frac{h}{\left[\sqrt{\left(\left(\inf T_{\tau_1}^i(h) \right)^2 + \left(\inf T_{\tau_2}^i(h) \right)^2 - \left(\inf T_{\tau_1}^i(h) \inf T_{\tau_2}^i(h) \right)^2}, \left(\sup T_{\tau_1}^i(h) \right)^2 + \left(\sup T_{\tau_2}^i(h) \right)^2 - \left(\sup T_{\tau_1}^i(h) \sup T_{\tau_2}^i(h) \right)^2 \right]}, \left[\left(\inf I_{\tau_1}^i(h) \inf I_{\tau_2}^i(h) \right), \left(\sup I_{\tau_1}^i(h) \sup I_{\tau_2}^i(h) \right) \right], \left[\left(\inf F_{\tau_1}^i(h) \inf F_{\tau_2}^i(h) \right), \left(\sup F_{\tau_1}^i(h) \sup F_{\tau_2}^i(h) \right) \right] \right\}$$

Where $h \in X$ and $i=1,2, \dots, m$.

Example 3.23. For τ_1 and τ_2 given Example 3.11, we have

$$\tau_1 \oplus \tau_2 = \left(\begin{array}{ccc} ([0.62,0.67], [0.18,0.6], [0.56,0.0]) & ([0.35,0.98], [0.18,0.6], [0.56,0.14]) & ([0.3,0.83], [0.20,0.18], [0.09,0.24]) \\ ([0.83,0.19], [0.0,0.27], [0.32,0.14]) & ([0.99,0.14], [0.12,0.63], [0.45,0.2]) & ([0.35,0.88], [0.48,0.24], [0.18,0.9]) \\ ([0.86,0.44], [0.12,0.18], [0.25,0.05]) & ([0.76,0.63], [0.02,0.08], [0.4,0.28]) & ([0.76,0.29], [0.72,0.14], [0.27,0.32]) \end{array} \right)$$

Definition 3.24. The product of two $PmIVFNS_s$ τ_1 and τ_2 take off the same universe X is explained as

$$\tau_1 \otimes \tau_2 = \left\{ \frac{h}{\left[\left(\left(\inf T_{\tau_1}^i(h) \inf T_{\tau_2}^i(h) \right), \left(\sup T_{\tau_1}^i(h) \sup T_{\tau_2}^i(h) \right) \right], \left[\left(\inf I_{\tau_1}^i(h) \inf I_{\tau_2}^i(h) \right), \left(\sup I_{\tau_1}^i(h) \sup I_{\tau_2}^i(h) \right) \right], \sqrt{\left[\left(\left(\inf F_{\tau_1}^i(h) \right)^2 + \left(\inf F_{\tau_2}^i(h) \right)^2 - \left(\inf F_{\tau_1}^i(h) \inf F_{\tau_2}^i(h) \right)^2, \left(\sup F_{\tau_1}^i(h) \right)^2 + \left(\sup F_{\tau_2}^i(h) \right)^2 - \left(\sup F_{\tau_1}^i(h) \sup F_{\tau_2}^i(h) \right)^2 \right]} \right] \right\}$$

For $h \in X$ and $i=1,2, \dots, m$.

Example 3.25. For τ_1 and τ_2 given in example 3.11, we have

$$\tau_1 \otimes \tau_2 = \left(\begin{array}{ccc} ([0.2,0.14], [0.18,0.6], [0.9,0.8]) & ([0.5,0.4], [0.18,0.6], [0.9,0.71]) & ([0.0,0.36], [0.2,0.18], [0.9,0.82]) \\ ([0.36,0.3], [0.0,0.27], [0.84,0.71]) & ([0.63,0.2], [0.12,0.63], [0.93,1.0]) & ([0.8,0.24], [0.48,0.24], [0.9,1.0]) \\ ([0.32,0.1], [0.12,0.18], [0.66,0.51]) & ([0.3,0.12], [0.2,0.8], [1.0,0.76]) & ([0.3,0.6], [0.72,0.14], [0.91,0.84]) \end{array} \right)$$

PYTHAGOREAN m-POLAR INTERVAL VALUED FUZZY NEUTROSOPHIC TOPOLOGY

In this section, we present Pythagorean m -polar interval valued fuzzy neutrosophic topology on Pythagorean m -polar interval valued fuzzy neutrosophic set and elongate numerous characteristics of crisp topology towards Pythagorean m -polar interval valued fuzzy neutrosophic topology. Separation axioms in $PmIVFNS_s$ are also discussed.

Definition 4.1. Let $PmIVFNS(X)$ be the collection of all $PmIVFN$ -subsets of the absolute $PmIVFNS$ X_A . For $S, T \subseteq A$, a subcollection $\mathfrak{P}_{pin}$ of $PmFNS(X)$ is known as *Pythagorean m -polar interval valued fuzzy neutrosophic topology* ($PmIVFNT$) on X if the following needs are satisfied:

- (i) $(\emptyset, X_A) \in \mathfrak{P}_{pin}$,
- (ii) $A, B \in \mathfrak{P}_{pin}$ then $A \cap B \in \mathfrak{P}_{pin}$
- (iii) $A_i \in \mathfrak{P}_{pin}, \forall i \in I$, then $\cup_{i \in I} A_i \in \mathfrak{P}_{pin}$.

The doublet $(X, \mathfrak{P}_{pin})$ or simply τ_{pin} , where X is a non – empty $PmIVFNS$ and $\mathfrak{P}_{pin}$ is a Pythagorean m -polar interval valued fuzzy neutrosophic topology on X , is known as *Pythagorean m -polar interval valued fuzzy neutrosophic topological space* ($PmIVFNS$).

Example 4.2. Let $X = \{h_1, h_2\}$ be a universal $P3IVFNS$ with S and T be as shown in table 4 and table5 below:

Table 4. $P3IVFNS(S)$

S	
h_1	$([0.2,1.0],[0.7,0.2],[0.6,0.4])$ $([0.3,0.1],[0.8,0.7],[0.6,0.1])$ $([0.5,0.3],[0.6,0.7],[0.3,0.4])$
h_2	$([0.2,0.7],[0.6,0.1],[0.5,0.6])$ $([0.2,0.4],[0.7,0.1],[0.2,0.5])$ $([0.1,0.3],[0.6,0.4],[0.8,0.3])$

Table 5. P3IVFNS (T)

T	
h_1	$([0.2,0.6],[0.5,0.2],[0.5,0.4]) ([0.1,0.5],[0.6,0.3],[0.7,0.4]) ([0.3,0.5],[0.6,0.1],[0.3,0.6])$
h_2	$([0.1,0.7],[0.6,0.1],[0.6,0.3]) ([0.5,0.7],[0.9,0.1],[0.7,0.2]) ([0.3,0.7],[0.6,0.4],[0.5,0.6])$

Then $\tau_{pin5} = \{\emptyset, S, T, X_A\}$ is a P3IVFNT on X.

Definition 4.3. The members of $\mathfrak{P}_{pin}$ are called *Pythagorean m-polar interval valued fuzzy neutrosophic open sets* (PmIVFN-open sets). The complements of Pythagorean m-polar interval valued fuzzy neutrosophic open sets are called *Pythagorean m-polar interval valued fuzzy neutrosophic closed sets* (PmIVFN-closed sets) and PmIVFN- open set as well as PmIVFN-closed set is called *Pythagorean m-polar interval valued fuzzy neutrosophic clopen sets* (PmIVFN- clopen sets).

Example 4.4. For the P3IVFNTS $\mathfrak{P}_{pin5}$ given in Example 4.2, we have $\emptyset, S, T, X_A$ are P3IVFN-open sets because they are members of $\mathfrak{P}_{pin5}$, $(X_A)^c = \emptyset \in \mathfrak{P}_{pin}$ is a P3IVFN- closed set and $\emptyset, X_A$ are P3IVFN-clopen sets as $\emptyset^c = X_A - \emptyset = X_A$ and $X_A^c = X_A - X_A = \emptyset$.

Example 4.5. Consider the P3IVFNSs X, S and T given in Example4.2 and

Table 6. P3IVFNS (U)

U	
h_1	$([0.2,0.4],[0.4,0.6],[0.5,0.2]) ([0.6,0.2],[0.1,0.5],[0.6,0.3]) ([0.3,0.5],[0.6,0.1],[0.3,0.7])$
h_2	$([0.8,0.1],[0.6,0.2],[0.3,0.6]) ([0.1,0.6],[0.8,0.4],[0.6,0.2]) ([0.5,0.6],[0.1,0.8],[0.5,0.2])$

We have ,

$$\begin{aligned}
 \mathfrak{P}_{pin1} &= \{\emptyset, X_A\} \\
 \mathfrak{P}_{pin2} &= \{\emptyset, S, X_A\} \\
 \mathfrak{P}_{pin3} &= \{\emptyset, T, X_A\} \\
 \mathfrak{P}_{pin4} &= \{\emptyset, U, X_A\} \\
 \mathfrak{P}_{pin5} &= \{\emptyset, S, T, X_A\} \\
 \mathfrak{P}_{pin6} &= \{\emptyset, T, U, X_A\} \\
 \mathfrak{P}_{pin7} &= \{\emptyset, S, U, X_A\} \\
 \mathfrak{P}_{pin8} &= \{\emptyset, S, T, U, X_A\}
 \end{aligned}$$

are Pythagorean 3-polar interval valued fuzzy neutrosophic topologies over X. Here, both $\emptyset$ & X_A are P3IVFN- open set as well as P3IVFN- closed set so it is a P3IVFN- clopen set.

Theorem 4.6. Let (X, τ_N) be an PmIVFN topological space. Then

- (i) $0_N, 1_N$ are PmIVFN – closed sets.
- (ii) Arbitrary intersection of PmIVFN – closed sets is PmIVFN – closed set.
- (iii) Finite union of PmIVFN – closed sets is PmIVFN – closed set.

Proof: (i) since $0_N, 1_N \in \tau_N$, $0_N^c = 1_N$ and $1_N^c = 0_N$ therefore 0_N^c and 1_N^c are PmIVFN – closed sets.

(ii) Let $\{A_i: i \in I\}$ be an arbitrary family of PmIVFN – closed sets in (X, τ_N) and let $A = \bigcap_{i \in I} A_i$. Now,

$$A^c = \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} (A_i)^c$$

and $A^c \in \tau_N$ for each $i \in I$, hence $\bigcup_{i \in I} (A_i)^c \in \tau_N$ therefore $A^c \in \tau_N$.

Thus A is an PmIVFN– closed set.

(iii) Let $\{A_k: k = 1, 2, \dots, n\}$ be a family of PmIVFN – closed set in (X, τ_N) and let $G = \bigcup_{k=1}^n A_k$. Now,

$$G^c = \left(\bigcup_{k=1}^n A_k \right)^c = \bigcap_{k=1}^n (A_k)^c$$

and $(A_K)^c \in \tau_N$ for $K=1,2,\dots,n$. So, $\bigcap_{K=1}^n A_K^c \in \tau_N$.
Hence, $G^c \in \tau_N$. Thus G is PmIVFN – Closed set.

Definition 4.7. Let $(X, \mathfrak{I}_{pin1})$ and $(X, \mathfrak{I}_{pin2})$ be two PmIVFNTSs on X . $\mathfrak{I}_{pin2}$ is contained in $\mathfrak{I}_{pin1}$ (i.e) $\mathfrak{I}_{pin2} \subseteq \mathfrak{I}_{pin1}$ if $k \in \mathfrak{I}_{pin2}$ for every $k \in \mathfrak{I}_{pin1}$. In such case, $\mathfrak{I}_{pin2}$ is known as *Pythagorean m -polar interval valued fuzzy neutrosophic coarser or weaker* (PmIVFN- coarser/weaker) than $\mathfrak{I}_{pin2}$ and $\mathfrak{I}_{pin1}$ is called *Pythagorean m -polar interval valued fuzzy neutrosophic finer or stronger* (PmIVFN-finer/stronger) than $\mathfrak{I}_{pin2}$. $\mathfrak{I}_{pin1}$ and $\mathfrak{I}_{pin2}$ in such a case are known as *comparable*. In Example 4.5, $\mathfrak{I}_{pin2}$ is PmIVFN-Coarser than $\mathfrak{I}_{pin5}$ and $\mathfrak{I}_{pin5}$ is PmIVFN-stronger than $\mathfrak{I}_{pin2}$. Hence $\mathfrak{I}_{pin2}$ and $\mathfrak{I}_{pin5}$ are comparable.

Definition 4.8. The PmIVFNT $\mathfrak{I}_{pin}(\text{indiscrete}) = \{\emptyset, X_A\}$ is known as *indiscrete Pythagorean m -polar interval valued fuzzy neutrosophic topology* (indiscrete-PmIVFNT) & $\mathfrak{I}_{pin}(\text{discrete}) = P(X_A)$ (powerset of X_A) is known as *discrete Pythagorean m -polar interval valued fuzzy neutrosophic topology* (discrete-PmIVFNT) over X .

Remark 4.9. On X , the smallest PmIVFNT is $\mathfrak{I}_{pin}(\text{indiscrete})$ whereas the largest PmIVFNT is $\mathfrak{I}_{pin}(\text{discrete})$.

Definition 4.10. Suppose that $(X, \mathfrak{I}_{pinX})$ be a PmIVFNTS. A few $Y \subseteq X$ and PmIVFN-open sets are $S_n^* = S_n \cap Y_A$ of PmIVFNT $\mathfrak{I}_{pinY}$ on Y where S_n are PmIVFN-open sets of $\mathfrak{I}_{pinX}$ & Y_A is absolute PmIVFNS on Y then $\mathfrak{I}_{pinY}$ is reserved as the *Pythagorean m -polar interval valued fuzzy neutrosophic subspace* (PmIVFN- subspace) of $\mathfrak{I}_{pinX}$. It can be written as:

$$\mathfrak{I}_{pinY} = \{S_n^*: S_n^* = S_n \cap Y_A, S_n \in \mathfrak{I}_{pinX}\}$$

Example 4.11. Let $\mathfrak{I}_{pinX} = \{\emptyset, S, T, X_A\}$, then $\mathfrak{I}_{pinX}$ is a P3IVFNT on X . $\mathfrak{I}_{pinX}$ is a P3IVFNS on $Y = \{S\} \subseteq X$ is

Table 7. P3IVFNS (Y_A)

Y_A	
h_1	$([1.0, 1.0], [0.0, 0.0], [0.0, 0.0]) ([1.0, 1.0], [0.0, 0.0], [0.0, 0.0]) ([1.0, 1.0], [0.0, 0.0], [0.0, 0.0])$

Since

$$\begin{aligned} \inf(Y_A \cap \emptyset), \sup(Y_A \cap \emptyset) &= \emptyset \\ \inf(Y_A \cap S), \sup(Y_A \cap S) &= \inf S, \sup S \\ \inf(Y_A \cap T), \sup(Y_A \cap T) &= \inf T, \sup T \\ \inf(Y_A \cap X_A), \sup(Y_A \cap X_A) &= \inf Y_A, \sup Y_A \end{aligned}$$

So, $\mathfrak{I}_{pinY} = \{\emptyset, S, T, Y_A\}$ is a *Pythagorean 3-polar interval valued fuzzy neutrosophic subtopology* (P3IVFN-subtopology) of $\mathfrak{I}_{pinX}$ (i.e $\mathfrak{I}_{pinY} \subseteq \mathfrak{I}_{pinX}$).

Remark 4.12. (1) A PmIVFN- subtopology i.e. $\mathfrak{I}_{pinZ}$ of a PmIVFN-subtopology $\mathfrak{I}_{pinY}$ of a PmIVFNTS $\mathfrak{I}_{pinX}$ is also a PmIVFN-subtopology of $\mathfrak{I}_{pinX}$.

(2) Every PmIVFN-subspace of a discrete-PmIVFNTS is always discrete-PmIVFNTS. Similarly, every PmIVFN-subspace of indiscrete-PmIVFNTS is also an indiscrete-PmIVFNTS.

Definition 4.13. Let $(X, \mathfrak{I}_{pin})$ be a PmIVFNTS and $V \subseteq PmIVFNS(X)$. The *Pythagorean m -polar interval valued fuzzy neutrosophic interior* (PmIVFN-interior) V_i of V is PmIVFNS which is the union of all PmIVFNS-open subsets (i.e that are contained in V) of X .

Theorem 4.14. Let (X, τ_N) be a PmIVFN topological space and $A, B \in PmIVFNS(X)$ then the following properties holds:

- (i) $Int(A) \subseteq A$
- (ii) $A \subseteq B \Rightarrow Int(A) \subseteq Int(B)$
- (iii) $Int(A) \in \tau_N$
- (iv) $A \in \tau_N \text{ iff } Int(A) = A$
- (v) $Int(Int(A)) = Int(A)$
- (vi) $Int(0_N) = 0_N, Int(1_N) = 1_N$

Proof:

- (i) Straight forward.
- (ii) $A \subseteq B \Rightarrow$ All of the PmIVFN open sets in A that are also in B . Both PmIVFN open sets included in A also included in B . i.e., $\{K \in \tau_N: K \subseteq A\} \subseteq \{G \in \tau_N: G \subseteq B\}$. i.e., $\cup \{K \in \tau_N: K \subseteq A\} \subseteq \cup \{G \in \tau_N: G \subseteq B\}$. i.e., $Int(A) \subseteq Int(B)$.

- (iii) $Int(A) = \bigcup \{K \in \tau_N : K \subseteq A\}$. It is clear that $\bigcup \{K \in \tau_N : K \subseteq A\} \in \tau_N$. So, $PmIVFN\ Int(A) \in \tau_N$.
- (iv) Let $A \in \tau_N$, then by (i), $Int(A) \subseteq A$. Now since $A \in \tau_N$ and $Int(A) \subseteq A$. Therefore $A \subseteq \bigcup \{G \in \tau_N : G \subseteq A\} = Int(A)$, $A \subseteq Int(A)$. Thus $Int(A) = A$. Conversely, let $Int(A) = A$. Since by (iii), $Int(A) \in \tau_N$. Therefore $A \in \tau_N$.
- (v) By (iii), $Int(A) \in \tau_N$. Therefore by (iv), $Int(Int(A)) = Int(A)$.
- (vi) We know that $0_N, 1_N \in \tau_N$, by (iv), $Int(0_N) = 0_N$, $Int(1_N) = 1_N$.

Definition 4.15. Let (X, τ_{pin}) be a PmIVFNTS and $V \subseteq PmIVFN(X)$. Then the *Pythagorean m-polar interval valued fuzzy neutrosophic closure* (PmIVFN-closure) $\dot{V}$ of V is the PmIVFNS which is intersection of all PmIVFN-closed supersets (i.e that contain V) of V .

Theorem 4.16. Let (X, τ_N) be a PmIVFNTS and $A, B \in PmIVFNTs(X)$ then possess the following properties:

- (i) $A \subseteq Cl(A)$
- (ii) $A \subseteq B \Rightarrow Cl(A) \subseteq Cl(B)$
- (iii) $(Cl(A))^c \in \tau_N$
- (iv) $A^c \in \tau_N$ iff $Cl(A) = A$
- (v) $Cl(Cl(A)) = Cl(A)$
- (vi) $Cl(0_N) = 0_N$, $Cl(1_N) = 1_N$

Proof:

Straight forward.

Theorem 4.17 Let (X, τ_N) be a PmIVFN topological space and $A, B \in PmIVFNTs(X)$ then hold the following properties:

- (i) $Int(A \cap B) = Int(A) \cap Int(B)$
- (ii) $Int(A \cup B) \supseteq Int(A) \cup Int(B)$
- (iii) $Cl(A \cup B) = Cl(A) \cup Int(B)$
- (iv) $Cl(A \cap B) \subseteq Cl(A) \cap Int(B)$
- (v) $(Int(A))^c = Cl(A^c)$
- (vi) $(Cl(A))^c = Int(A^c)$

Proof. (i) By Theorem 4.14(i), $Int(A) \subseteq A$ and $Int(B) \subseteq B$. Thus $Int(A) \cap Int(B) \subseteq A \cap B$. Hence $Int(A) \cap Int(B) \subseteq Int(A \cap B)$ -----(1)

Again since $A \cap B \subseteq A$, by Theorem 4.14(ii). $Int(A \cap B) \subseteq Int(A)$.

Similarly, $Int(A \cap B) \subseteq Int(B)$.

Hence $Int(A \cap B) \subseteq Int(A) \cap Int(B)$ -----(2)

from (1) and (2) we get,

$$Int(A \cap B) = Int(A) \cap Int(B).$$

(ii) Since $A \subseteq A \cup B$. $Int(A) \subseteq Int(A \cup B)$ by Theorem 4.14(ii).

Similarly $Int(B) \subseteq Int(A \cup B)$. Hence $Int(A) \cup Int(B) \subseteq Int(A \cup B)$.

(iii) By Theorem 4.16(i), $A \subseteq Cl(A)$ and $B \subseteq Cl(B)$. Thus $A \cup B \subseteq Cl(A) \cup Cl(B)$, $Cl(A \cup B) \subseteq Cl(A) \cup Cl(B)$ -----(1)

Again since $A \subseteq A \cup B$, by Theorem 4.16(ii). $Cl(A) \subseteq Cl(A \cup B)$.

Similarly $Cl(B) \subseteq Cl(A \cup B)$

Hence $Cl(A) \cup Cl(B) \subseteq Cl(A \cup B)$ -----(2)

from (1) and (2) we get $Cl(A) \cup Cl(B) = Cl(A \cup B)$.

Since $A \cap B \subseteq A$, $Cl(A \cap B) \subseteq Cl(A)$ by Theorem 4.16(ii), Similarly, $Cl(A \cap B) \subseteq Cl(B)$. Hence $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$.

(iv) $\{Int(A)\}^c = [\bigcup \{G \in \tau_N : G \subseteq A\}]^c = \bigcap \{G \in \tau_N^c : A^c \subseteq G\},$

$$\{Int(A)\}^c = Cl(A)^c.$$

$$(v) \quad \{Cl(A)\}^c = [\cap \{G \in \tau_N^c : A^c \subseteq G\}]^c = \cup \{G \in \tau_N : G \subseteq A\},$$

$$\{Cl(A)\}^c = Int(A)^c.$$

In theorem 4.17(ii) and (iv)), the equality does not hold.

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