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A Novel Type I Half-Logistic Exponentiated Inverse Exponential Distribution Applications To Covid-19 Data Set

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Abstract : The aim of this paper is to develop a new flexible statistical model to examine the COVID-19 data sets that cannot be modeled by a classical inverse exponential distribution. A novel extended distribution with one scale and two shape parameters is proposed using the type 1 Half-Logistic Exponentiated distribution. Some important properties of the new distribution like Probability Weighted Moments**,** moments, Moment Generating Function, Reliability function, Quantile Function, expressions of order statistics are derived. Parameters of the derived distribution are obtained using maximum likelihood method. The performances of the estimators are assessed through Monte-Carlo simulation, which shows that the maximum likelihood method works well in estimating the parameters. The new distribution was applied to COVID-19 data sets in order access the flexibility and adaptability of the distribution. The result shows that new distribution performance better than other comparators models.

Keywords: Probability Weighted Moments**,** moments, Moment Generating Function, Reliability function, Quantile Function, order statistics, Maximum Likelihood.

I. Introduction

The creation of novel, generalized statistical models is a significant field of research in distribution theory. Such distributions, which are particularly useful in forecasting and simulating real-world phenomena, are widely available in the literature. Over the past few decades, a variety of classical distributions have been extensively utilized to model data in a variety of practical fields, such as biomedical analysis, reliability engineering, economics, forecasting, astronomy, demography, and insurance.

Significant work has gone into creating common probability distributions and the associated statistical techniques. Classical distributions frequently do not offer an appropriate match to real-world data sets. As a result, various families of continuous distributions have been constructed in the literature, using one or more parameters to produce new distributions. These generalized classes of distributions can accommodate both monotonic and non-monotonic hazard failure rate shapes and are particularly adaptable. Many of the developed families were examined in various contexts and found to produce more application versatility. For example, Alzaatreh et al., (2013) defined the T-X family of distribution, Yousof et al., (2018), obtained a generalized version of the Marshall-Olkin-G family of distributions, Nasir et al., (2017) proposed a new family of distributions called the generalized Burr-G family of distributions, Alizadeh et al., (2015) introduced the Kumaraswamy Marshall-Olkin family of distributions, Afify et al., (2016) defined the Kumaraswamy Transmuted-G family of distributions, Silva et al., (2019) defined a new class of distributions that extends the Kumaraswamy-G family of distributions, Al-Shomrani et al., (2016) proposed the Topp-Leone-G family of distributions, Yousof et al., (2017) defined a new family of distributions called transmuted Topp-Leone-G family of distributions, Sanusi, et al., (2020) proposed a new family of distributions called Toppleone Exponential-G family of distributions, Falgore and Doguwa (2020) proposed a new generator of continuous distributions with four positive parameters called the Kumaraswamy-Odd Rayleigh-G family of distributions, Ibrahim et al., (2020a) developed Topp-Leone exponentiated-G family of distributions, Bello et al., (2020a) proposed Type I half logistic exponentiated-G family of distributions, Bello et al., (2020b) proposed Type II half logistic exponentiated-G family of distributions.

The positive half of the logistic distribution is represented by a probability distribution called the half logistic distribution. It has a symmetric, bell-shaped curve that is bounded at zero and is a continuous probability distribution. The distribution is frequently used in probability theory and statistics to describe positive continuous variables. When working with positive data that displays a symmetric pattern and has a natural bottom bound at zero, the half logistic distribution is frequently employed as a modeling tool. It can be used in disciplines like dependability analysis, finance, and economics.

The half logistic-G distribution can be used to model a variety of data, including survival times, waiting times, and economic data. It has been shown to be more flexible than the half logistic distribution, and it can be used to model data with a wider range of shapes. Using the half-logistic distribution, Cordeiro et al., (2017) introduced the type I half-logistic family of distributions, Adepoju et al., (2021) introduced the type I half logistic-T-G family of distributions, Moakofi et al., (2021) derived type II exponentiated half-logistic Topp-Leone Marshall-Olkin-G family of distributions.

2.0 NEED OF THE STUDY.

The main objective this paper is to developed a novel model that more robust and flexible in modeling COVID-19 data set and also to evaluate the impact of introducing additional parameters to the distribution and how this affects its flexibility, applicability, and overall effectiveness. By comparing the baseline distribution with the new distributions that have additional parameters, we can gain insights into how these modifications enhance or alter the distribution's ability to fit COVID-19 data and address various modeling challenges.

3.0 METHODOLOGY

3.1 Type I Half-Logistic Exponentiated-G Family of Distributions

Bello *et al.* (2021) introduced the Type I Half-Logistic Exponentiated-G Family of distributions, which offers enhanced flexibility for modeling real-life datasets. The cumulative distribution function (cdf) of the Type I Half-Logistic Exponentiated-G Family of distributions, as defined by Bello *et al.* (2021), is provided.

$$F_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{1-[1-H^{\alpha}(x;\boldsymbol{\beta})]^{\lambda}}{1+[1-H^{\alpha}(x;\boldsymbol{\beta})]^{\lambda}},\; x>0,\lambda,\alpha>0 \text{ and } \boldsymbol{\beta} \text{ is parameter vector} \quad (1)$$

Where λ and α are shape parameters, H(x;$\boldsymbol{\beta}$) is the cdf of the baseline model. The corresponding probability distribution function (pdf) is given by

$$f_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{2\lambda\alpha h(x;\boldsymbol{\beta})H^{\alpha-1}_{(x;\boldsymbol{\beta})}[1-H^{\alpha}_{(x;\boldsymbol{\beta})}]^{\lambda-1}}{\left[1+[1-H^{\alpha}_{(x;\boldsymbol{\beta})}]^{\lambda}\right]^{2}},\; x>0,\lambda,\alpha>0 \quad (2)$$

3.2 Inverse Exponential Distribution

Suppose that X is a continuous random variable, then X is said to have followed inverse exponential distribution if it cdf and pdf are expressed as;

$$H(x)=e^{-\left(\frac{\theta}{x}\right)} \quad (3)$$

$$h(x)=\left(\frac{\theta}{x^2}\right)e^{-\left(\frac{\theta}{x}\right)}, \quad x>0,\theta>0 \tag{4}$$

where θ is a scale parameter

The random variable X is said to have a TIHLEtIEx distribution, if its cdf is obtained by inserting equation (3) in equation (1) as follows

$$F(x,\alpha,\lambda,\theta)=\frac{1-\left[1-\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha}\right]^{\lambda}}, x>0,\ \lambda,\alpha \text{ and } \theta>0 \tag{5}$$

On differentiating (5), the pdf of TIHLEtIEx is as follows

$$f(x,\alpha,\lambda,\theta)=\frac{2\alpha\lambda\left(\frac{\theta}{x^2}\right)e^{-\left(\frac{\theta}{x}\right)}\left[e^{-\frac{\theta}{x}}\right]^{\alpha-1}\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}\right]^{2}} \tag{6}$$

3.3 Expansion of Density for TIHLEtIEx Distribution

In this section, a useful expansion of the PDF and CDF for TIHLEtIEx Distribution are provided. Since the generalized binomial series is

$$[1+z]^{-b}=\sum_{i=0}^{\infty}(-1)^i\binom{b+i-1}{i}z^i \tag{7}$$

Using the last term in the equation (6) in relation to equation (7), we have

$$\left[1+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}\right]^{-2}=\sum_{i=0}^{\infty}(-1)^i\binom{1+i}{i}\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda i} \tag{8}$$

Substituting equation (8) into equation (6), we have

$$f(x,\alpha,\lambda,\theta)=2\alpha\lambda\left(\frac{\theta}{x^2}\right)e^{-\left(\frac{\theta}{x}\right)}\left[e^{-\frac{\theta}{x}}\right]^{\alpha-1}\sum_{i=0}^{\infty}(-1)^i\binom{1+i}{i}\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda(i+1)-1} \tag{9}$$

On expanding the last term in equation (9), we have

$$\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda(i+1)-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\lambda(i+1)-1}{j}\left[e^{-\frac{\theta}{x}}\right]^{\alpha j} \qquad (10)$$

Substituting equations (10) into equation (9), we have

$$f(x,\alpha,\lambda,\theta)=2\alpha\lambda\left(\frac{\theta}{x^{2}}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i}(-1)^{j}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha(j+1)} \qquad (11)$$

The equation (11) above can be rewrite, so we have

$$f(x,\alpha,\lambda,\theta)=\sum_{i,j=0}^{\infty}\gamma\frac{\theta}{x^{2}}\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha(j+1)} \qquad (12)$$

where $\gamma=2\alpha\lambda(-1)^{i+j}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}$

Equation (12) is the important representation of the pdf of TIHLEtIEx distribution.

Also, an expansion for the CDF, using the binomial expansion $\left[F(x,\alpha,\lambda,\theta)\right]^{h}$ where h is an integer, lead to:

$$\left[F(x,\alpha,\lambda,\theta)\right]^{h}=\underbrace{\left[1-\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}\right]^{h}}_{A}\underbrace{\left[1+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}\right]^{-h}}_{B} \qquad (13)$$

Using the A term in the equation (13) in relation to equation (7), we have

$$A=\left[1-\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}\right]^{h}=\sum_{p=0}^{h}(-1)^{p}\binom{h}{p}\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda p} \qquad (14)$$

Also using the B term in the equation (13) in relation to equation (7), we have

$$B=\left[1+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}\right]^{-h}=\sum_{k=0}^{h}(-1)^{k}\binom{h+k-1}{k}\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda k} \qquad (15)$$

Combining equation (14) and equation (15), we obtain

$$\left[F(x,\alpha,\lambda,\theta)\right]^{h}=\sum_{p=0}^{h}\sum_{k=0}^{h}(-1)^{p}(-1)^{k}\binom{h}{p}\binom{h+k-1}{k}\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda(p+k)} \qquad (16)$$

On expanding the last term in equation (16), we have

$$\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda(p+k)}=\sum_{z=0}^{\infty}(-1)^{z}\binom{\lambda(p+k)}{z}\left[e^{-\frac{\theta}{x}}\right]^{\alpha z} \quad (17)$$

Substituting equations (17) into equation (16), we have

$$\left[F(x,\alpha,\lambda,\theta)\right]^{h}=\sum_{z=0}^{\infty}\sum_{p,k=0}^{h}(-1)^{p+k+z}\binom{h}{p}\binom{h+k-1}{k}\binom{\lambda(p+k)}{z}\left[e^{-\frac{\theta}{x}}\right]^{\alpha z} \quad (18)$$

The expression above can be rewrite, so we have

$$\left[F(x,\alpha,\lambda,\theta)\right]^{h}=\sum_{p,k=0}^{h}\xi\left[e^{-\frac{\theta}{x}}\right]^{\alpha z} \quad (19)$$

where $\xi=\sum_{z=0}^{\infty}(-1)^{p+k+z}\binom{h}{p}\binom{h+k-1}{k}\binom{\lambda(k+p)}{z}$

Now (19) is the important representation of the cdf of TIHLEtIEx distribution.

3.4 *Statistical Properties of TIHLEtIEx Distribution*

This section explores various statistical properties of TIHLEtIEx distribution.

3.4.1 *Probability Weighted Moments of the TIHLEtIEx Distribution*

Greenwood *et al.* (1979) introduced a class of moments known as probability weighted moments (PWMs). This class is used to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, represented by $\tau_{r,s}$ can be derived for a random variable X using the following relationship.

$$\tau_{r,s}=E\left[X^{r}F(X)^{s}\right]=\int_{-\infty}^{\infty}x^{r}f(x)(F(x))^{s}dx \quad (20)$$

The PWMs of TIHLEtIEx is derive by substituting equation (12) and equation (19) into equation (20) replacing h with s, we have

$$\tau_{r,s}=\sum_{i,j=0}^{\infty}\sum_{p,k=0}^{s}\gamma\xi\frac{\theta}{x^{2}}\int_{0}^{\infty}x^{r}\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+z+1)}dx \quad (21)$$

Consider the integral part in equation (21)

$$\int_{0}^{\infty}x^{r}\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+z+1)}dx \quad (22)$$

Let $y=\alpha(j+z+1)\left(\frac{\theta}{x}\right)$; x $=x=\alpha(j+z+1)\left(\frac{\theta}{y}\right)$ and $dx=\frac{dyx^{2}}{\alpha\theta(j+z+1)}$

$$\int_0^\infty \left[\alpha(j+z+1)\left(\frac{\theta}{y}\right)\right]^r e^{-y}\frac{dyx^2}{\alpha\theta(j+z+1)} \tag{23}$$

Substituting equations (23) into equation (21), we have

$$\tau_{r,s}=\sum_{i,j=0}^{\infty}\sum_{p,k=0}^{s}\gamma\xi\frac{\theta}{x^2}\int_0^\infty \left[\alpha(j+z+1)\left(\frac{\theta}{y}\right)\right]^r e^{-y}\frac{dyx^2}{\alpha\theta(j+z+1)} \tag{24}$$

$$\tau_{r,s}=\sum_{i,j=0}^{\infty}\sum_{p,k=0}^{s}\gamma\xi\alpha^r\theta^r(j+z+1)^{r-1}\int_0^\infty y^{-r}e^{-y}dy \tag{25}$$

where $\int_0^\infty y^{-r}e^{-y}dy=\Gamma(1-r)$

Therefore

$$\tau_{r,s}=\sum_{i,j=0}^{\infty}\sum_{p,k=0}^{s}\gamma\xi\alpha^r\theta^r(j+z+1)^{r-1}\Gamma(1-r) \tag{26}$$

Now $\gamma=2\lambda(-1)^{i+j}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}$ and $\xi=\sum_{z=0}^{\infty}(-1)^{p+k+z}\binom{s}{p}\binom{s+k-1}{k}\binom{\lambda(k+p)}{z}$

The equation (26) above is the PWMs of TIHLEtIEx.

3.4.2 Moments of the TIHLEtIEx Distribution

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the r^th moments for TIHLEtIEx distribution.

$$E(X^r)=\int_0^\infty x^r f(x)dx \tag{27}$$

The r^th moments for TIHLEtIEx distribution is derive by substituting equation (12) into equation (27) we obtain

$$E(X^r)=\sum_{i,j=0}^{\infty}\gamma\frac{\theta}{x^2}\int_0^\infty x^r\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+1)}dx \tag{28}$$

Consider the integral part of equation (28)

$$\int_0^\infty x^r\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+1)}dx \tag{29}$$

Let $y=\alpha(j+1)\left(\frac{\theta}{x}\right)$; $x=\alpha(j+1)\left(\frac{\theta}{y}\right)$ and $dx=\frac{dyx^2}{\alpha\theta(j+1)}$

$$\int_0^{\infty}\left[\alpha(j+1)\left(\frac{\theta}{y}\right)\right]^r e^{-y}\frac{dyx^2}{\alpha\theta(j+1)} \tag{30}$$

Substituting equations (30) into equation (28), we have

$$E\left[X^r\right]=\sum_{i,j=0}^{\infty}\gamma\frac{\theta}{x^2}\int_0^{\infty}\left[\alpha(j+1)\left(\frac{\theta}{y}\right)\right]^r e^{-y}\frac{dyx^2}{\alpha\theta(j+1)} \tag{31}$$

$$E\left[X^r\right]=\sum_{i,j=0}^{\infty}\gamma\alpha^r\theta^r(j+1)^{r-1}\int_0^{\infty}y^{-r}e^{-y}dy \tag{32}$$

where $\int_0^{\infty}y^{-r}e^{-y}dy=\Gamma(1-r)$

So therefore

$$E\left[X^r\right]=\sum_{i,j=0}^{\infty}\gamma\alpha^r\theta^r(j+1)^{r-1}\Gamma(1-r) \tag{33}$$

Now, $\gamma=2\lambda\left(-1\right)^{i+j}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}$

The equation (33) above is the r^{th} moments for TIHLEtIEx distribution and the mean of the distribution will be obtained by setting r = 1 in (30)."

3.4.3 Moment Generating Function (mgf) of TIHLEtIEx Distribution

The Moment Generating Function of x is given as

$$M_x(t)=\int_0^{\infty}e^{tx}f(x)dx \tag{34}$$

The mgf for TIHLEtIEx distribution is derive by substituting equation (12) into equation (34) we obtain

$$M_x(t)=\sum_{i,j=0}^{\infty}\gamma\frac{\theta}{x^2}\int_0^{\infty}e^{tx}\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+1)}dx \tag{35}$$

where the expansion of $e^{tx}=\sum_{z=0}^{\infty}\frac{t^z x^z}{z!}$ and following the process of moments above, we have the mgf for TIHLEtIEx distribution in equation (36) below.

$$M_x(t)=\sum_{i,j=0}^{\infty}\sum_{m=0}^{\infty}\gamma\frac{t^m}{m!}\alpha^m\theta^m(j+1)^{m-1}\Gamma(1-m) \qquad (36)$$

3.4.4 Reliability function of TIHLEtIEx Distribution

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as:

$$R(x;\lambda,\alpha,\theta)=1-F(x;\lambda,\alpha,\theta) \qquad (37)$$

$$R(x;\lambda,\alpha,\theta)=\frac{2\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}} \qquad (38)$$

3.4.5 Hazard Function of TIHLEtIEx Distribution

The hazard function provides information about how the risk of an event changes over time. It characterizes the failure rate or the rate of occurrence of the event at a particular time, conditional on the event not occurring before that time. It can be defined as

$$T(x;\lambda,\alpha,\theta)=\frac{f(x;\lambda,\alpha,\theta)}{R(x;\lambda,\alpha,\theta)} \qquad (39)$$

$$T(x;\lambda,\alpha,\theta)=\frac{\alpha\lambda\frac{\theta}{x^2}e^{-\frac{\theta}{x}}\left[e^{-\frac{\theta}{x}}\right]^{\alpha-1}}{\left[1+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]\right]\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]} \qquad (40)$$

3.4.6 Quantile Function of TIHLEtIEx Distribution

The quantile function which is also known as inverse CDF, of the TIHLEtIEx distribution is obtained by using the CDF in equation (5).

$$F(x;\lambda,\alpha,\theta)=\frac{1-\left[1-\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha}\right]^{\lambda}}=U$$

$$1-\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}=U\left[1+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}\right]$$

$$1-\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}=U+U\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}$$

$$1-U=U\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}+\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}$$

$$1-U=U+1\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}$$

$$\left[1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}\right]^{\lambda}=\frac{1-U}{U+1}$$

$$1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha}=\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}$$

$$\left[e^{-\frac{\theta}{x}}\right]^{\alpha}=1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}$$

$$e^{-\frac{\theta}{x}}=\left[1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}$$

$$\frac{-\theta}{x}=\log\left[1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}$$

$$x=Q(u)=\frac{\theta}{\left[-\log\left[1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]} \tag{41}$$

3.4.7 Distribution of Order Statistics of TIHLEtIEx Distribution

Let $X_1, X_2,..., X_3$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function F(x). Let $X_{1:n}<X_{2:n}<...<X_{n:n}$ the corresponding ordered random sample fro

m the TIHLEtIEx distributions. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, r=1,2,3,...,n denote the CDF and PDF of the r^{th} order statistics $X_{r:n}$ respectively. The PDF of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x;\lambda,\alpha,\theta)=\frac{f(x)}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}\left[F(x)\right]^{v+r-1} \tag{42}$$

The PDF of r^{th} order statistic for TIHLEtIEx distribution is derived by substituting equation (12) and equation (19) into equation (42). Also replacing h with v+r-1 in equation (19), so we have

$$f_{r:n}(x;\lambda,\alpha,\theta)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i,j=0}^{\infty}\sum_{p,k=0}^{r+v-1}(-1)^{v}\binom{n-r}{v}\gamma\xi\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+z+1)} \tag{43}$$

Now

$$\xi=\sum_{z=0}^{\infty}(-1)^{p+k+z}\binom{\mathrm{v+r-1}}{p}\binom{\mathrm{v+r}+k-2}{k}\binom{\lambda(k+p)}{z}$$

and

$$\gamma=2\alpha\lambda\frac{\theta}{x^{2}}(-1)^{i+j}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}$$

The PDF of minimum order statistic of the TIHLEtIEx distribution is obtained by setting r = 1 in equation (43) as

$$f_{1:n}(x;\lambda,\alpha,\theta)=2\alpha\lambda n\frac{\theta}{x^{2}}\sum_{v}^{n-1}\sum_{i,j=0}^{\infty}\sum_{z=0}^{\infty}\sum_{p,k=0}^{v}(-1)^{v+i+j+z+p+k}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}\binom{v}{p}\binom{v+k-1}{k}\binom{\lambda(k+p)}{z}\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+z+1)} \tag{44}$$

Also, the PDF of maximum order statistic of the TIHLEtIEx distribution is obtained by setting r = n in equation (44) as

$$f_{n:n}(x;\lambda,\alpha,\theta)=2\alpha\lambda n\frac{\theta}{x^{2}}\sum_{i,j=0}^{\infty}\sum_{z=0}^{\infty}\sum_{p,k=0}^{n+v-1}(-1)^{v+i+j+z+p+k}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}\binom{v+n-1}{p}\binom{\mathrm{v}+n+k-2}{k}\binom{\lambda(k+p)}{z}\left[e^{-\frac{\theta}{x}}\right]^{\alpha(j+z+1)} \tag{45}$$

3.5 *Parameter Estimation of TIHLEtIEx Distribution*

In this section, we will derive the method of estimation to estimate the unknown parameters of the TIHLEtIEx distribution. Here, we consider two methods for the estimation of the unknown parameters of the TIHLEtIEx distribution.

3.5.1 *Maximum Likelihood Estimation of TIHLEtIEx Distribution*

Let $x_1, x_2, x_3, ..., x_n$ be a random sample of size n from the TIHLEtIEx distribution. Then, the likelihood function based on observed sample for the vector of parameter $(\lambda, \alpha, \theta)^T$ is given

$$\log(L) = n\log(2) + n\log(\alpha) + n\log(\lambda) + n\log(\theta) + \sum_{i=1}^{n}\log\left(\frac{1}{x_i^2}\right) - \sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha} + (\lambda-1)\sum_{i=1}^{n}\log\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right] - 2\sum_{i=1}^{n}\log\left[1+\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]^{\lambda}\right] \tag{46}$$

Differentiating the log-likelihood with respect to λ, α, θ and equating the result to zero, we have

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha}\log\left(\frac{\theta}{x_i}\right) - (\lambda-1)\sum_{i=1}^{n}\frac{\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\log\left[e^{-\frac{\theta}{x_i}}\right]}{1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}} + 2\lambda\sum_{i=1}^{n}\frac{\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]^{\lambda-1}\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\log\left[e^{-\frac{\theta}{x_i}}\right]}{1+\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]^{\lambda}} = 0 \tag{47}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right] - 2\sum_{i=1}^{n}\frac{\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]^{\lambda}\log\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]}{1+\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]^{\lambda}} = 0 \tag{48}$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} - \alpha\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha-1}\frac{1}{x_i} + \alpha(\lambda-1)\sum_{i=1}^{n}\frac{\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha-1}e^{-\frac{\theta}{x_i}}\frac{1}{x_i}}{1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}} + 2\alpha\lambda\sum_{i=1}^{n}\frac{\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]^{\lambda-1}\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha-1}e^{-\frac{\theta}{x_i}}\frac{1}{x_i}}{1+\left[1-\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\right]^{\lambda}} = 0 \tag{49}$$

The equation (47), (48) and (49) above are non linear, cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

4.0. RESULTS AND DISCUSSION

4.1 Simulation Study

In this section, a simulation study was conducted to evaluate the consistency of parameter estimates obtained using Maximum Likelihood Estimation (MLE) for the newly introduced model. The purpose was to assess how well the estimated parameters align with the true values through the simulation analysis.

4.1.2 Simulation Study of Type I Half-Logistic Exponentiated Inverse Exponential (TIHLEtIEx) Distribution

In this study, 1000 replicates were generated from the TIHLEtIEx distribution using the quantile function defined in equation (3.44). The sample sizes chosen were n=20, 50, 100, 150, 200, and 250. The resulting replicates were used to compute parameter estimates, bias, and Root Mean Square Error (RMSE), which are presented in Tables 1 provide the MLE estimates, as well as the corresponding bias and RMSE for the estimated parameters of TIHLEtIEx at specific values of λ = 0.4, α =1, θ=0.5, and λ = 1, α =0.5, θ=2.1, respectively. The tables demonstrate that the biases and RMSEs approach zero, indicating that the estimates become more accurate as the sample size increases. This suggests that the estimates obtained are both efficient and consistent.

Table1: MLEs, biases and RMSE for some values of parameters

		(0.4,1,0.5)			(1,0.5,2.1)		
N	Parameters	Estimated Values	Bias	RMSE	Estimated Values	Bias	RMSE
20	λ	0.4466	0.0466	0.1203	1.0000	0.0000	0.0000
	α	1.1042	0.1042	0.1930	0.6185	0.1185	0.1484
	θ	0.6156	0.1156	0.3363	2.1406	0.0406	0.0747
50	λ	0.4180	0.0180	0.0636	1.0000	0.0000	0.0000 0.
	α	1.0647	0.0647	0.1117	0.629	0.129	1520
	θ	0.5305	0.0305	0.1812	2.143	0.043	0.0735

100	λ	0.4083	0.0083	0.0403	1.0000	0.0000	0.0000
	α	1.0498	0.0498	0.0829	0.6155	0.1155	0.1413
	θ	0.5109	0.0109	0.1283	2.1335	0.0335	0.0648
150	λ	0.4047	0.0047	0.0327	1.0000	0.0000	0.0000
	α	1.0385	0.0385	0.0685	0.6032	0.1032	0.1332
	θ	0.5023	0.0023	0.1015	2.1348	0.0348	0.0612
200	λ	0.4041	0.0041	0.0362	1.0000	0.0000	0.0000
	α	1.0353	0.0353	0.0560	0.6015	0.1015	0.1319
	θ	0.5016	0.0016	0.0759	2.1324	0.0324	0.0580
250	λ	0.4033	0.0033	0.0318	1.0000	0.0000	0.0000
	α	1.0345	0.0345	0.0541	0.6003	0.1003	0.1215
	θ	0.5010	0.0010	0.0673	2.1272	0.0272	0.0504

4.2 Application of the new models to Real-Life Data sets

Data Set 1

The first data set as listed below represents the daily confirmed cases of COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by Al-Marzouki, *et al.*, (2020):

108, 102, 133, 170, 121, 99, 236, 178, 250, 161, 258, 172, 407, 577, 210, 243, 281, 186, 254, 336, 342,269, 520, 414, 463, 514, 427, 796, 555, 742, 642, 785, 783, 605, 751, 806.

4.2.1 Fitting Type I Half-Logistic Exponentiated Inverse Exponential Distribution

In this section, we applied the TIHLEtIEx distribution to analyze both data set 1. We conducted a comprehensive comparative study by fitting the TIHLEtIE distribution against other distributions like ToLIEx, EtIEx, EtIEx, and KIEx. The purpose of this comparison was to illustrate the versatility and suitability of the new distribution and determine how well it fits the experimental data sets compared to the comparator distributions. Maximum Likelihood Estimate (MLE) techniques were utilized in the analysis. By using the R programming language, we carried out all the computations, making the entire process efficient and straightforward.

4.2.2 The Comparators

The pdf of the comparators considered are:

- Topp-Leone Inverse Exponential (ToLIEx) Distribution

$$f(x;\theta,\beta)=2\theta\left(\frac{\beta}{x^2}\right)e^{-\frac{\beta}{x}}\left[1-e^{-\frac{\beta}{x}}\right]\left[1-\left[1-e^{-\frac{\beta}{x}}\right]^2\right]^{\theta-1}$$

- Exponentiated Inverse Exponential (EIEx) Distribution

$$f(x;\alpha,\beta)=\frac{\alpha\beta}{x^2}\left(e^{-\frac{\beta}{x}}\right)^{\alpha}$$

- Inverse Exponential (IEx) Distribution

$$f(x;\beta)=\frac{\beta}{x^2}e^{-\frac{\beta}{x}}$$

- Kumaraswamy Inverse Exponential (KIEx) Distribution

$$f(x;\alpha,\lambda,\beta)=\alpha\lambda\left(\frac{\beta}{x^2}\right)e^{-\frac{\alpha\beta}{x}}\left[1-e^{-\frac{\alpha\beta}{x}}\right]^{\lambda-1}$$

4.2.3 Results with Comparators

In this section, we conduct a comparison using the baseline inverse exponential distribution as a reference point. The purpose is to assess how the introduction of additional parameters to the distribution affects its flexibility, applicability, and effectiveness.

Table 2: The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on daily confirmed cases of COVID-19 (Data set 1)

Model	α	θ	λ	β	LL	AIC
TIHLEtIEx	4.1705	1.7442	3.3429	-	-243.2150	492.4299
ToLIEx	3.2955	2.1158	-	-	-245.0929	494.1858
EIEx	6.9821	-	-	6.9148	-250.8568	505.7137
IEx	-	-	-	1.9522	-267.4646	536.9292
KIEx	7.1368	-	7.8649	3.0254	-243.5948	493.1897

The table 2. displays the outcomes of the Maximum Likelihood Estimation for the parameters of the TIHLEtIEx distribution and four other comparable distributions. By examining the goodness of fit measure, it was observed that the TIHLEtIEx distribution achieved the lowest AIC value of 492.4299. As a result, among all the considered distributions, the TIHLEtIEx distribution demonstrated the most favorable fit to the daily confirmed COVID-19 dataset.

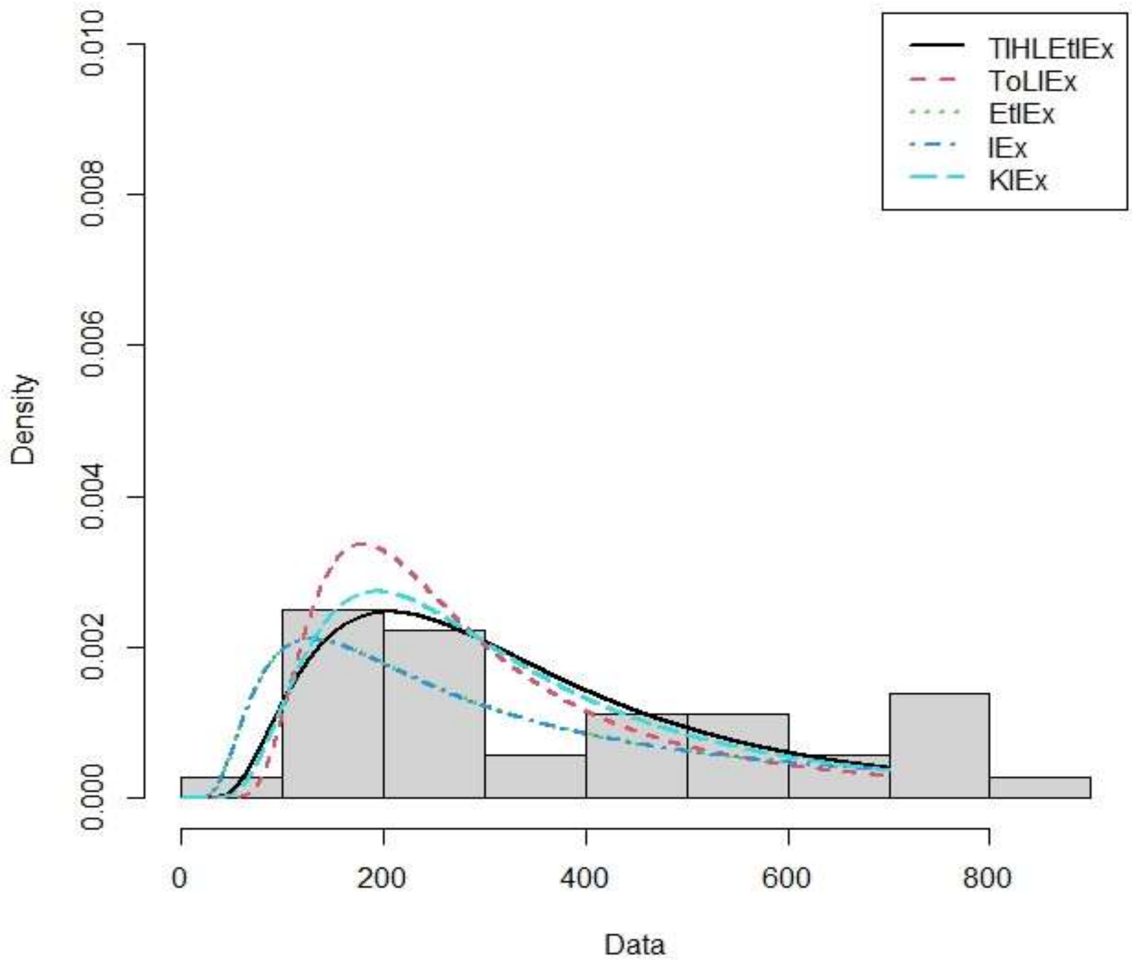

Figure 1: Fitted pdfs for the TIHLEtIEx, ToLIEx, EIEx, IEx, and KIEx models to the data set 1.

Figure 1 provides a visual representation of the fit of the TIHLEtIEx distribution and its comparator distributions. Upon visual inspection, it becomes evident that the TIHLEtIEx distribution outperforms the other distributions in terms of fitting the data. This visual confirmation further supports the conclusion that the TIHLEtIEx distribution is superior to its comparator distributions in accurately describing the dataset of interest.

5.0 CONCLUSION

In this paper, a comparative analysis is performed using the baseline inverse exponential distribution as a reference point. The main objective is to evaluate the impact of introducing additional parameters to the distribution and how this affects its flexibility, applicability, and overall effectiveness. By comparing the baseline distribution with the new distributions that have additional parameters, we can gain insights into how these modifications enhance or alter the distribution's ability to fit real-life data and address various modeling challenges. The results of simulation study shows that the biases and RMSEs approach zero, indicating that the estimates become more accurate as the sample size increases. This suggests that the estimates obtained are both efficient and consistent.

The results of AIC showed that the TIIHLEtIEx distribution obtained the lowest AIC value as compared with others comparators distributions models. Also, the result of the goodness of fit shows that TIIHLEtIEx distribution is the most favorable fit to COVID-19 data as compare to other comparator models.

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