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On Concavity of Fuzzy sets

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Abstract:
The main purpose of this paper is to study the concavity of fuzzy sets defined on the R and Cartesian product of the set of real numbers respectively. We use the concept of fuzzy epigraph to prove these results.

Keywords:
Fuzzy sets, Concave Fuzzy sets, Concave Fuzzy Relations, α−level sets, Fuzzy Epigraph.

I. INTRODUCTION

Convexity is a fundamental idea that is studied in many pure and applied science ideas. Since concavity is the opposite of convexity, understanding concavity in functions is also important. The convexity of these functions makes it easier to study and isolate their features. In 1965, Prof. Zadeh first coined the terms fuzzy set, convex fuzzy set, fuzzy relation, and α-cut. Fuzzy set □ is a function □: □ → *[0, 1]*, called as membership function and □(□) is called membership grade at □ in □; □ ∈ □; values of membership grades lie in [0,1]. Fuzzy relation M is a fuzzy set defined on the Cartesian product of the set of real numbers. Concave fuzzy sets were first conceptualized by B.B. Choudhuri [10] during his investigation of various fuzzy set forms.By Yu-Ru Syau [11], the concept was extended to convex and concave fuzzy mappings. In addition to introducing concavo-convex fuzzy sets, Sarkar [12] demonstrated other fascinating features of this specific type of fuzzy set. Convex temporal intuitionistic fuzzy sets were developed and convex intuitionistic fuzzy sets were insightfully discussed by Ban [13, 14]. Díiaz et al. [15] conducted a thorough analysis and characterisation of the generalized aspects of the aggregation of convex intuitionistic fuzzy sets.

Researchers Syau [5] and Xinmin Yang [2] presented closed and convex fuzzy sets and examined their relationship. The convex fuzzy relation was distinguished by Nadaban and Dzitac[4], who also provided instances of particular fuzzy relation types in their study. Based on previous work, Chen-Wei-Xu[6] developed new fuzzy relations and provided convexity conclusions for fuzzy relations. In [10], Complementary α−level sets of fuzzy sets are defined and shown with the circular and elliptic holes as concave fuzzy sets . The fuzzy set F is concave,iff membership grade of point t on the line segment joining □ □□□ □ is less than or equal to the maximum of membership grade of □ □□□ □ ∀ □, □ □ □$^{□}$. This paper will investigate its properties.We will extend the concavity of a fuzzy Set with respect to □□□□□□□□□□□□□□ □ −level set to Fuzzy Relation and try to prove that fuzzy relation T is concave then complementary □ −level set of the fuzzy relation, □$_{□}$ is convex; ∀□ ∈ (0, 1].We will study that □□□□□□□□□□□□□□ □ − level set of a fuzzy set is an interval if F is a concave fuzzy set and study its connection with a strongly concave fuzzy set and strictly concave fuzzy set.

II.PRELIMINERIES

Throughout this paper, B denotes fuzzy set defined on M denotes fuzzy relation defined on R^2 Here are some definitions that will be useful in this paper.

2.1Definition[6]:

A fuzzy set B defined on R is a function; $B: R \rightarrow [0,1]$ is called as membership function and $B(x)$ is called membership grade of $B \; at \; x$.

2.2Definition[7]:

A Fuzzy relation M is a fuzzy set defined on Cartesian product of crisp sets $Y_1 \times Y_2 \times Y_3 \times \ldots\ldots \times Y_n$ where tuples $(y_1, y_2, y_3, \ldots\ldots, y_n)$ that may have varying degrees of membership value is usually represented by a real number for closed intervals[0,1] and indicate the strength of the present relation between elements of the topic.

Consider $M: X \times Y \rightarrow [0,1]$ then the fuzzy relation on $X \times Y$ denoted by M or $M(x, y)$is defined as the set $M(X,Y) = \{((x,y), S(x,y))/(x,y) \in X \times Y\}$. where $M(x,y)$ is the strength of the relation in two variables called membership function. It gives the degree of membership of the ordered pair $(x,y) \; in \; X \times Y$ a real number in the interval [0,1].

2.3Definition[3]:

M be fuzzy relation on $X \times Y$. Then T is concave if and only if $M\big(\mu(x_1, y_1) + (1-\mu)(x_2, y_2)\big) \leq$ $Max[M(x_1, y_1) \wedge M(x_2, y_2)\,]$; $\forall \,(x_1, y_1), (x_2, y_2) \in X \times Y \; and \;\; \mu \in [0,1]$.

2.4Definition[6]:

Let M be a fuzzy relation defined on $X \times Y$ and α be such that $0 < \alpha \leq$ 1.Then □□□□□□□□□□□□□□ α −level set of M is denoted by $M_\alpha -$ is defined by $M_\alpha - = \{(x,y) \in X \times Y / T(x,y) \leq \; \alpha\}$.

2.5 Definition [1]:

B be a fuzzy set defined on R and α be such that $0 < \alpha \leq 1$.Then α −level of B, is denoted by B_α and defined by $B_\alpha = \{x \in R / B(x) \geq \alpha\}$ is a crisp set.

2.6 Definition [1]:

B be a fuzzy set defined on R. Then M is concave if and only if

$B(\mu x_1 + (1-\mu)x_2) \leq max[B(x_1), B(x_2)]$; $\forall \; x_1, x_2 \in R \; and \; \mu \in (0, 1]$.

2.7 Definition[2]:

A fuzzy set B on R is said to be strongly convex fuzzy set if

$B(\mu x_1 + (1-\mu)x_2) < max[B(x_1), B(x_2)]$; $\forall\, x_1, x_2 \in R, x_1 \neq x_2$ and $\mu \in (0,1)$.

2.8 Definition [2]:

A fuzzy set B on is said to be strictly concave fuzzy set if

$B(\mu x_1 + (1-\mu)x_2) < max[B(x_1), B(x_2)]$; $B(x_1) \neq B(x_2), \forall x_1, x_2 \in R$ *and* $\mu \in (0,1)$.

III. MAIN RESULTS

3.1 Theorem

B be a fuzzy set defined on R then M is concave then if for every $\alpha \in (0, 1]$,

□□□□□□□□□□□□□ α −level set; $B_\alpha -$ is hole/ non convex.

Proof:

Suppose M is a concave fuzzy set defined on R.

to prove that $B_\alpha -$is non convex, $\forall\alpha \in (0, 1]$.

Let, if possible, for some $\alpha \in (0, 1]$. $B_\alpha -$ is convex.

That is there exist $x, y \in B_\alpha$ such that,

$\mu x + (1-\mu)y \in B_\alpha$; $\forall\mu \in [0,1]$.

Implies that, $B(\mu x + (1-\mu)y) \geq \alpha$

Since, $x \in B_\alpha, B(x) \geq \alpha$.

and $y \in B_\alpha, B(y) \geq \alpha$.

Given M is a concave fuzzy set.

Therefore $B(\mu x + (1-\mu)y) \leq max(B(x), B(y))$

$\leq max(\alpha, \alpha)$.

$= \alpha$.

By definition of □□□□□□□□□□□□□ α −level set of fuzzy set,

$\mu x + (1-\mu)y \notin B_\alpha$.

A contradiction to our assumption that B_α is convex, for some $\alpha \in (0, 1]$.

Therefore B_α is convex, $\forall\alpha \in (0, 1]$.

3.2 Corollary:

B be a strongly (strictly) concave fuzzy set defined on R then $B_\alpha -$ is not convex; for all $\alpha \in (0, 1]$, where $B_\alpha -$ is strong *complementary* α −cut of B.

3.3 Theorem

M be a fuzzy relation defined on $R \times R$ then M is concave then for every $\alpha \in (0, 1]$;$M_\alpha -$ is not convex.

Proof.

Suppose that M is concave fuzzy relation defined on $R \times R$.

to prove that $M_\alpha -$ is not convex.

Let if possible, for some $\alpha \in (0, 1]$. $M_\alpha -$ is convex.
Then $\exists (x_1, y_1), (x_2, y_2) \in M_\alpha -$ such that
$\mu(x_1, y_1) + (1-\mu)(x_2, y_2) \in T_\alpha -; \ \mu \in [0,1]$.
Implies that $M\big(\mu(x_1, y_1) + (1-\mu)(x_2, y_2)\big) \leq \alpha$.
Since, $(x_1, y_1) \in M_\alpha -; M(x_1, y_1) \leq \alpha$. (1)
and $(x_2, y_2) \in M_\alpha -; \ M(x_2, y_2) \leq \alpha$.
Since M is concave fuzzy relation.
Consider,
$M\big(\mu(x_1, y_1) + (1-\mu)(x_2, y_2)\big) \leq max(M(x_1, y_1), M(x_2, y_2))$.
$\leq min(\alpha, \alpha)$. (by 1)
$= \alpha$.
By definition of $complementary \ \alpha$ −level set of fuzzy relation,
$\mu(x_1, y_1) + (1-\mu)(x_2, y_2) \in M_\alpha -$.
A contradiction.
M_α- is non convex set, for every $\alpha \in (0, 1]$.

3.4 Theorem

M be a strongly concave fuzzy relation defined on R^2 then there exist unique element (x_1, x_2) such that $M(x_1, x_2) = min\{M(y_1, y_2)/(y_1, y_2) \in R^2\}$.

Proof.

Suppose that there are two elements (x_1, x_2) and (z_1, z_2) in R^2such that

$\alpha = M(x_1, x_2) = M(z_1, z_2) = min\{M(y_1, y_2)/(y_1, y_2) \in R^2\}$.

where, $(x_1, x_2) \neq (z_1, z_2)$.

$M_\alpha - = \{(x_1, x_2),(z_1, z_2)\}$.

Given that M is strongly concave fuzzy relation on R^2. therefore $M_\alpha -$ is non convex; $\alpha \in (0, 1)$.

As $(x_1, x_2) \neq (z_1, z_2)$.

We have, $\mu(x_1, x_2) + (1-\mu)(z_1, z_2) \notin M_\alpha -$as $M_\alpha -$ contains only two elements; for all $\alpha \in (0, 1)$ and $\mu \in (0,1)$.

Therefore $M_\alpha -$ is not convex. A contradiction.

$\therefore (x_1, x_2) = (z_1, z_2)$.

Therefore, there exist unique element(x_1, x_2) such that $S(x_1, x_2) = min\{M(y_1, y_2)/(y_1, y_2) \in R^2\}$

Conversely, part of the above theorem is not true in general; we may take non concave fuzzy set and find a unique minimum element.

3.5 Theorem

M is concave fuzzy relation defined on R^2 then for every $\alpha \in (0, 1]$, M_α- is neither closed nor connected.

Proof.

First, we prove that $M_\alpha -$ is not a closed set.

suppose that it is closed.

Let (x, y) be any limit point of $M_\alpha -$ then there exists sequence$(x_n, y_n) \in M_\alpha -$.

$(x_n, y_n) \to (x, y)\ \forall\ n \geq N$..

As $(x_n, y_n) \in M_\alpha -$ we have $M(x_n, y_n)) < \alpha$. Therefore □□□ □ $\to \infty\ M(x_n, y_n) < \alpha$.

(x, y) is limit point of $M_\alpha -$; (x, y) is either an interior point or boundary point of $M_\alpha -$.

Case 1:(x, y) is an interior point of $M_\alpha -$.then clearly $M(x, y) \leq \alpha$.

Case 2:(x, y) is boundary point of $M_\alpha -$. We can find $\in > 0$ such that $(x-\in, x+\in)$ is an interior point of $M_\alpha -$.

Given that M is concave fuzzy relation on R^2.

Then by definition of concavity of fuzzy relation, $M_\alpha -$ is not convex, for all $\alpha \in (0, 1]$.

For,(x_n, y_n), (x-$\in, x+\in$) $\in M_\alpha -$; $\mu(x_n, y_n) + (1-\mu)(x-\in, x+\in) \notin M_\alpha -$.

$$\lim \mu(x_n, y_n) + (1-\mu)(x-\in, x+\in) \notin M_\alpha -.$$

$\lim \mu(x_n, y_n) + lim\ (1-\mu)(x-\in, x+\in) \notin M_\alpha -$.

$\mu(x, y) + (1-\mu)(x-\in, y-\in) \notin M_\alpha -$.

$for\ \mu = 1, (x, y) \notin M_\alpha -$.

Therefore $M(x, y) \geq \alpha$.

Since (x, y) is an arbitrary limit point of $M_\alpha -$.

Therefore $M_\alpha -$ is not closed.

Now to prove that $M_\alpha -$ is not connected.

Let, if possible, it is connected. Then there does not exist two non-empty, disjoint, open sets A and B such that $M_\alpha -$ =A∗B and A)B=∅.

We can choose $(x, y) \in A\ and\ (x', y') \in B$.

Then $\mu(x, y) + (1-\mu)(x', y') \in M_\alpha -$.

$M_\alpha -$ is a convex set. Contradiction. Therefore $M_\alpha -$ is not connected.

Conversely, the above theorem is not true in general.

For example, $M(x, y) = 1\ if\ (x, y) \in \left\{(x, y) \in R^2\ \ /x^2 + y^2 \geq 4\ and\ x^2 + y^2 \leq 9\right\}$.

$$= \frac{1}{2}\ \ if\ (x, y) \in ⍰R^2/x^2 + y^2 < 1\ and\ x^2 + y^2 > 4\}.$$

Then $M_1 -= (x, y) \in \{(x, y) \in R^2/\ x^2 + y^2 < 1\ and\ x^2 + y^2 > 4\}$.

which is not closed and not connected but concave.

Fuzzy Epigraph

Fuzzy epigraph of a fuzzy set M is denoted by □. □□□(□) = $\{$(□, □)$: x \in R^n, t \in (0, M(x)]\}$.

3.6 Theorem

A fuzzy set $B: R^n \to [0,1]$ is connected then its fuzzy epigraph; □. □□□(□) $= \{($□, □$): x \in R^n, t \in (0, B(x)]\}$ **is c**onnected subset of $R^n X(0,1]$.

Proof:Suppose that fuzzy set $B: R^n \to [0,1]$ is connected.

That means every complementary $\alpha -$ □□□□□ □□□ is not connected; for all $\alpha\epsilon(0,1]$

To prove that □. □□□(□) is connected.

Let if possible □. □□□(□) is not connected then there exist two non-empty,open,disjoint sets □ □□□ □ □□□h □h□□ □ $\cup$ □ $\neq$ □. □□□(□) □□□ □ $\cap$ □ $\neq$ □.

Case1. If □ □□□ □ are not convex then by the hyperplane separation theorem of convex sets there exists a hyperplane Υ which separates them. let α' be the level set which passes through Υ then complementary $\alpha' -$ □□□□□ □□□ = □ $\cup$ □\{□} is a disjoint union of two closed sets;where □ □□□ □ □□□ □□□□□□□ □□ □□□□□□□□□□□□□ □$'$ − □□□□□ □□□ . which makes $\alpha' -$ □□□□□ □□□ not connected. A contradiction to B is a connected fuzzy set.

Case 2.If □ □□□ □ are convex. since P and Q are disjoint, open and B is subset of □. □□□(□) . Some part of the fuzzy set B is contained in C and another part of B is contained in Q.

Without loss of generality assume that P and Q are disconnected at a point (□$_1$, □(□$_1$)) of fuzzy set B. Take any □□□□□□□□□□□□□ □$'$ − □□□□□ □□□ ; where $\alpha' \geq$ □(□$_1$). □$_□$ − is a union of at least two disjoint sets. □$_□$ − is disconnected. Contradiction to assumption to that B is connected fuzzy set defined on Therefore B is connected subset of $R^n X(0,1]$.

IV. RESULTS AND DISCUSSION

I.Concavity of fuzzy sets and concave fuzzy relations have been investigated with the help of □□□□□□□□□□□□□ □ − □□□□□ □□□ .The relationship between the fuzzy epigraph and the concave fuzzy set (concave fuzzy relation) was shown. Between fuzzy and crisp sets, the complimentary α-level set serves as a link. It is essential to study concavity using a fuzzy approach on multiple levels because of its enormous applications in many different domains.

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