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A Study on Differential Equations (Exactness)

K.Jyothi
Assistant Professor
Department of Mathematics
Nishitha Degree College(Autonomous)
Affiliated to Telangana University,Nizamabad
Accredited with "A" Grade by NAAC

Abstract:

A differential equation is an equation that contains one or more functions with its derivatives. The derivatives of the function define the rate of change of a function at a point. It is mainly used in fields such as physics, engineering, biology and so on. The primary purpose of the differential equation is the study of solutions that satisfy the equations and the properties of the solutions. Learn how to solve differential equations here.

One of the easiest ways to solve the differential equation is by using explicit formulas. In this article, let us discuss the definition, types, methods to solve the differential equation, order and degree of the differential equation, ordinary differential equations with real-word examples and a solved problem.

Key words: differential, methods, primary, physics, engineering, etc.,

Introduction:

Differential Equation applications have significance in both academic and real life.

An equation denotes the relation between two quantity or two functions or two variables or set of variables or between two functions. Differential equation denotes the relationship between a function and its derivatives, with some set of formulas. There are many examples, which signifies the use of these equations.

The functions are the one which denotes some sort of operation performed and the rate of change during the performance is the derivative of that operation, and the relation between them is the differential equation. These equations are represented in the form of order of the degree, such as first order, second order, etc. Its applications are common to find in the field of engineering, physics etc.

Significance of differential equations:

- Differential equations are important in mathematics and the sciences because they can be used to model a wide variety of real-world situations.
- In physics, for example, differential equations can be used to model the motion of particles in a fluid or the trajectory of a projectile.
- In biology, differential equations can be used to model the growth of populations or the spread of diseases.
- The ability to model complex situations using differential equations makes them a valuable tool for scientists and engineers.
- By solving a differential equation, they can gain a better understanding of how a system behaves and how it might be manipulated to achieve a desired outcome.

- Additionally, differential equations can be used to predict the future behavior of a system, which can be helpful in designing new technologies or predicting the outcomes of experiments.

Important in real life:

- Differential equations have a remarkable ability to predict the world around us.
- They are used in a wide variety of disciplines, from biology, economics, physics, chemistry and engineering.
- They can describe exponential growth and decay, the population growth of species or the change in investment return over time.

Types of differential equations

Basically, there are two types of differential equations

1.Ordinary Differential Equation (ODE)

Ordinary differential equation involves a relation between one real variable which is independent say x and one dependent variable say y and sum of derivatives y', y'', y'''… with respect to the value of x.

f(x) = y =d(y)/d(x)

The highest derivative which occurs in the equation is the order of ordinary differential equation. ODE for nth order can be written as;

$F(x,y,y',\ldots,y^n) = 0$

2.Partial differential equation

In mathematics, a partial differential equation (PDE) is an equation which imposes relations between the various partial derivatives of a multivariable function.

The function is often thought of as an "unknown" to be solved for, similarly to how *x* is thought of as an unknown number to be solved for in an algebraic equation like $x^2 - 3x + 2 = 0$. However, it is usually impossible to write down explicit formulas for solutions of partial differential equations. There is, correspondingly, a vast amount of modern mathematical and scientific research on methods to numerically approximate solutions of certain partial differential equations using computers. Partial differential equations also occupy a large sector of pure mathematical research, in which the usual questions are, broadly speaking, on the identification of general qualitative features of solutions of various partial differential equations, such as existence, uniqueness, regularity, and stability.

In this article, we are going to discuss what is an exact differential equation, standard form, integrating factor, and how to solve exact differential equation in detail with examples and solved problems.

Ordinary Differential Equations

Exact Differential Equation Definition

The equation P(x,y) dx + Q (x,y) dy=0 is an exact differential equation if there exists a function f of two variables x and y having continuous partial derivatives such that the exact differential equation definition is separated as follows

$u_x(x, y) = p(x, y)$ and $u_y(x, y) = Q(x, y)$

Therefore, the general solution of the equation is u(x, y) = C.

Where C is an arbitrary constant.

Testing for Exactness

Assume the functions P(x, y) and Q(x, y) having the continuous partial derivatives in a particular domain D, and the differential equation is exact if and only if it satisfies the condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Exact Differential Equation Integrating Factor

If the differential equation P (x, y) dx + Q (x, y) dy = 0 is not exact, it is possible to make it exact by multiplying using a relevant factor u(x, y) which is known as integrating factor for the given differential equation.

Consider an example,

1.2ydx + x dy = 0

Now check it whether the given differential equation is exact using testing for exactness.

The given differential equation is not exact.

In order to convert it into the exact differential equation, multiply by the integrating factor u(x,y)= x, the differential equation becomes,

2 xy dx + x^2 dy = 0

The above resultant equation is exact differential equation because the left side of the equation is a total differential of x^2y.

Sometimes it is difficult to find the integrating factor. But, there are two classes of differential equation whose integrating factor may be found easily. Those equations have the integrating factor having the functions of either x alone or y alone.

When you consider the differential equation P (x,y) dx + Q (x,y) dy=0, the two cases involved are:

Case 1: If $[1/Q(x,y)][P_y(x, y) – Q_x(x,y)] = h(x)$, which is a function of x alone, then $e^{\int h(x)dx}$ is an integrating factor.

Case 2: If $[1/P(x,y)][Q_x(x, y) – P_y(x,y)] = k(y)$, which is a function of y alone, then $e^{\int k(y)dy}$ is an integrating factor.

How to Solve Exact Differential Equation

The following steps explains how to solve the exact differential equation in a detailed way.

Step 1: The first step to solve exact differential equation is that to make sure with the given differential equation is exact using testing for exactness.

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Step 2: Write the system of two differential equations that defines the function u(x,y). That is

$$\frac{\partial u}{\partial x} = P(x, y)$$

$$\frac{\partial u}{\partial y} = Q(x, y)$$

Step 3: Integrating the first equation over the variable x , we get

$$u(x,y) = \int P(x,y)dx + \varphi(y)$$

Instead of an arbitrary constant C, write an unknown function of y.

Step 4: Differentiating with respect to y, substitute the function u(x,y) in the second equation

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x,y)dx + \varphi(y)\right] = Q(x,y)$$

From the above expression we get the derivative of the unknown function φ (y) and it is given by

$$\varphi(y) = Q(x,y) - \frac{\partial}{\partial y}\left(\int P(x,y)dx\right)$$

Step 5: We can find the function φ (y) by integrating the last expression so that the function u(x,y) becomes

$$u(x,y) = \int P(x,y)dx + \varphi(y)$$

Step 6: Finally, the general solution of the exact differential equation is given by

u (x,y) = C.

Exact Differential Equation Problems

Question: Find the solution for the differential equation (2xy – sin x) dx + (x^2 – cos y) dy = 0

Solution:

Given, (2xy – sin x) dx + (x^2 – cos y) dy = 0

First check this equation for exactness,

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - \cos y) = 2x$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy - \sin x) = 2x$$

The equation is exact because it satisfies the condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

From the system of two equations, find the functions u (x, y)

$$\frac{\partial u}{\partial x} = 2xy - \sin x \ldots.(1)$$

$$\frac{\partial u}{\partial y} = x^2 - \cos y \ldots.(2)$$

By integrating the first equation with respect to the variable x, we get

$$u(x,y) = \int (2xy - \sin x)dx = x^2 + \cos x + \varphi(y)$$

Substituting the above equation in equation (2) , it becomes

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^2 y + \cos x + \varphi(y)] = x^2 - \cos y$$

$$\Rightarrow x^2 + \varphi(y) = x^2 - cosy$$

We get

$$\Rightarrow \varphi(y) = -cosy$$

Hence,

$$\varphi(y) = \int (-\cos y)dy = -\sin y$$

So the function u (x , y) becomes

$u(x, y) = x^2y + \cos x - \sin y$

Therefore the general solution for the given differential equation is

$x^2y + \cos x - \sin y = C$

Non Exact Differential equations

Integrating Factors

If a differential equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (*)$$

is not exact as written, then there exists a function μ(*x,y*) such that the equivalent equation obtained by multiplying both sides of (*) by μ,

$$(\mu M)\,dx + (\mu N)\,dy = 0$$

is exact. Such a function μ is called an integrating factor of the original equation and is guaranteed to exist if the given differential equation actually has a solution. *Integrating factors turn nonexact equations into exact ones*. The question is, how do you find an integrating factor? Two special cases will be considered.

- *Case 1:*

Consider the differential equation $M\,dx + N\,dy = 0$. If this equation is not exact, then M_y will not equal N_x; that is, $M_y - N_x \neq 0$. However, if

$$\frac{M_y - N_x}{N}$$

is a function of *x* only, let it be denoted by ξ(*x*). Then

$$\mu(x) = e^{\int \xi(x)\,dx}$$

will be an integrating factor of the given differential equation.

- *Case 2:*

Consider the differential equation $M\,dx + N\,dy = 0$. If this equation is not exact, then M_y will not equal N_x; that is, $M_y - N_x \neq 0. = 0$. However, if

$$\frac{M_y - N_x}{-M}$$

is a function of y only, let it be denoted by $\psi(y)$. Then

$$\mu(y) = e^{\int \psi(y)\,dy}$$

will be an integrating factor of the given differential equation.

Example 1: The equation

$$(3xy - y^2)\,dx + x(x - y)\,dy = 0$$

is not exact, since

$$M_y = \frac{\partial}{\partial y}(3xy - y^2) = 3x - 2y \quad \text{but} \quad N_x = \frac{\partial}{\partial x}(x^2 - xy) = 2x - y$$

However, note that

$$\frac{M_y - N_x}{N} = \frac{(3x - 2y) - (2x - y)}{x(x - y)} = \frac{x - y}{x(x - y)} = \frac{1}{x}$$

is a function of x alone. Therefore, by Case 1,

$$e^{\int (1/x)\,dx} = e^{\ln x} = x$$

will be an integrating factor of the differential equation. Multiplying both sides of the given equation by $\mu = x$ yields

$$\underbrace{(3x^2y - xy^2)}_{\mu M = \overline{M}}dx + \underbrace{(x^3 - x^2y)}_{\mu N = \overline{N}}\,dy = 0$$

which *is* exact because

$$\frac{\partial \overline{M}}{\partial y} = 3x^2 - 2xy = \frac{\partial \overline{N}}{\partial x}$$

Solving this equivalent exact equation by the method described in the previous section, M is integrated with respect to x,

$$\int \overline{M}\,\partial x = \int (3x^2y - xy^2)\partial x = x^3y - \tfrac{1}{2}x^2y^2$$

and N integrated with respect to y:

$$\int \overline{N}\,\partial y = \int (x^3 - x^2y)\partial y = x^3y - \tfrac{1}{2}x^2y^2$$

(with each "constant" of integration ignored, as usual). These calculations clearly give

$$x^3y - \tfrac{1}{2}x^2y^2 = c$$

as the general solution of the differential equation.

Example 2: The equation

$$(x + y)\sin y\, dx + (x \sin y + \cos y)\, dy = 0$$

is not exact, since

$$M_y = (x + y)\cos y + \sin y \quad \text{but} \quad N_x = \sin y$$

However, note that

$$\frac{M_y - N_x}{-M} = \frac{(x + y)\cos y + \sin y - \sin y}{-(x + y)\sin y} = -\frac{\cos y}{\sin y}$$

is a function of *y* alone (Case 2). Denote this function by $\psi(y)$; since

$$\int \psi(y)\, dy = -\int \frac{\cos y\, dy}{\sin y} = -\ln(\sin y)$$

the given differential equation will have

$$e^{\int \psi(y)\, dy} = e^{-\ln(\sin y)} = e^{\ln(\sin y)^{-1}} = (\sin y)^{-1}$$

as an integrating factor. Multiplying the differential equation through by $\mu = (\sin y)^{-1}$ yields

$$\underbrace{(x + y)\, dx}_{\mu M = \overline{M}} + \underbrace{\left(x + \frac{\cos y}{\sin y}\right)}_{\mu N = \overline{N}} dy = 0$$

which *is* exact because

$$\overline{M}_y = 1 = \overline{N}_x$$

To solve this exact equation, integrate *M* with respect to *x* and integrate *N* with respect to *y*, ignoring the "constant" of integration in each case:

$$\int \overline{M}\, \partial x = \int (x + y)\, \partial x = \tfrac{1}{2}x^2 + xy$$

$$\int \overline{N}\, \partial y = \int \left(x + \frac{\cos y}{\sin y}\right) \partial y = xy + \ln|\sin y|$$

These integrations imply that

$$\tfrac{1}{2}x^2 + xy + \ln|\sin y| = c$$

is the general solution of the differential equation.

Example 3: Solve the IVP

$$(3e^x y + x)\,dx + e^x\,dy = 0$$
$$y(0) = 1$$

The given differential equation is not exact, since

$$M_y = \frac{\partial}{\partial y}(3e^x y + x) = 3e^x \quad \text{but} \quad N_x = \frac{\partial}{\partial x}(e^x) = e^x$$

However, note that

$$\frac{M_y - N_x}{N} = \frac{3e^x - e^x}{e^x} = 2$$

$$e^{\int \xi(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

will be an integrating factor. Multiplying both sides of the differential equation by $\mu(x) = e^{2x}$ yields

$$\underbrace{(3e^{3x}y + xe^{2x})}_{\mu M = \overline{M}}\,dx + \underbrace{(e^{3x})}_{\mu N = \overline{N}}\,dy = 0$$

which *is* exact because

$$\overline{M}_y = 3e^{3x} = \overline{N}_x$$

Now, since

$$\int \overline{M}\,\partial x = \int (3e^{3x}y + xe^{2x})\,\partial x = e^{3x}y + \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}$$

and

$$\int \overline{N}\,\partial y = \int e^{3x}\,\partial y = e^{3x}y$$

(with the "constant" of integration suppressed in each calculation)

The general solution of the differential equation is

$$e^{3x}y + \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} = c$$

The value of the constant c is now determined by applying the initial condition $y(0) = 1$:

$$[e^{3x}y + \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}]_{x=0,\,y=1} = c \Rightarrow \tfrac{3}{4} = c$$

Thus, the particular solution is

$$e^{3x}y + \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} = \tfrac{3}{4}$$

which can be expressed explicitly as

$$y = \frac{3e^{-3x} + e^{-x}(1 - 2x)}{4}$$

Example 4: Given that the nonexact differential equation

$$(5xy^2 - 2y)\,dx + (3x^2y - x)\,dy = 0$$

has an integrating factor of the form $\mu(x,y) = x^a y^b$ for some positive integers a and b, find the general solution of the equation.

Since there exist positive integers a and b such that $x^a y^b$ is an integrating factor, multiplying the differential equation through by this expression must yield an exact equation. That is,

$$\underbrace{(5x^{a+1}y^{b+2} - 2x^a y^{b+1})}_{\mu M = \overline{M}}\,dx + \underbrace{(3x^{a+2}y^{b+1} - x^{a+1}y^b)}_{\mu N = \overline{N}}\,dy = 0 \quad (*)$$

is exact for some a and b. Exactness of this equation means

$$\overline{M}_y = \overline{N}_x$$

$$5(b+2)x^{a+1}y^{b+1} - 2(b+1)x^a y^b = 3(a+2)x^{a+1}y^{b+1} - (a+1)x^a y^b$$

By equating like terms in this last equation, it must be the case that

$$5(b+2) = 3(a+2) \quad \text{and} \quad 2(b+1) = a+1$$

The simultaneous solution of these equations is $a = 3$ and $b = 1$.

Thus the integrating factor $x^a y^b$ is $x^3 y$, and the exact equation M dx + N dy = 0 reads

$$(5x^4y^3 - 2x^3y^2)\,dx + (3x^5y^2 - x^4y)\,dy = 0$$

Now, since

$$\int \overline{M}\partial x = \int (5x^4y^3 - 2x^3y^2)\partial x = x^5y^3 - \tfrac{1}{2}x^4y^2$$

and

$$\int \overline{N}\partial y = \int (3x^5y^2 - x^4y)\partial y = x^5y^3 - \tfrac{1}{2}x^4y^2$$

(ignoring the "constant" of integration in each case), the general solution of the differential equation (*)—and hence the original differential equation—is clearly

$$x^5y^3 - \tfrac{1}{2}x^4y^2 = c$$

Conclusion:

Ordinary differential equations have several applications and are a potent tool in the study of a variety of problems in the natural sciences and technology; they are widely used in machines, astronomy, physics, and a variety of chemistry and biology problems.

Exact equation, is a type of differential equation that can be solved directly without the use of any of the special techniques in the subject. A first-order differential equation (of one variable) is called exact, or an exact differential, if it is the result of a simple differentiation.

Linear differential equations stand out among ordinary differential equations for various reasons. The majority of basic and special functions encountered in physics and applied mathematics are linear differential equation solutions. The few non-linear ODEs that can be solved explicitly are usually solved by converting the equation to a linear ODE counterpart. Numerical approaches for ordinary differential equations can provide an estimate of the solution for applicable solution. A mathematical equation having only one independent variable and one or more derivates affecting that variable is known as "Ordinary Differential equation.

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