



# Calculating Ricci Curvature Tensor and Ricci scalar for the Traversable Wormhole

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## **Abstract**

A wormhole can be visualized as a tunnel with two ends at separate points in space-time (i.e., different locations, different points in time, or both). If the two ends of the wormhole lie in the same universe, it is called an intra-universe wormhole, while if the two ends are situated in two different universes, it is called an inter-universe wormhole. Traversability has considered an essential feature of a wormhole. In a more general way, if an entity can enter from one side of the wormhole and depart from the other, the wormhole is traversable. Morris and Thorne assumed a traversable wormhole that is spherically symmetrical, time-independent, and non-rotating. In this project, I will derive the mathematical equations for the Ricci Curvature Tensor ( $R_0$ ,  $R_1$ ,  $R_2$ , &  $R_3$ ) and Ricci Scalar ( $R$ ) for the traversable Wormhole using the Morris and Thorne metric.

## Introduction

General relativity is a theory of gravitation that was developed by Albert Einstein between 1907 and 1915. According to general relativity, the observed gravitational effect between masses results from their warping of space-time.

In general relativity, the world line of a particle-free from all external, non-gravitational force is a particular type of geodesic in curved space-time. In other words, a freely moving or falling particle always moves along a geodesic.

The geodesic equation is:

$$d^2x^\mu/d\tau^2 + \Gamma^\mu_{ab} (dx^a/d\tau) (dx^b/d\tau) = 0$$

where 'τ' is a scalar parameter of motion like the proper time, and  $\Gamma^\mu_{ab}$  is the Christoffel symbol sometimes called the affine connection coefficients or Levi-Civita connection coefficients which are symmetric in the two lower indices.

Where, indices μ, a, and b varies from 0 to 3.

Here, Coordinates of the space-time are taken as  $(x^0, x^1, x^2, x^3)$

Where,  $x^0$  denotes the time coordinate and  $x^1, x^2$ , and  $x^3$  denote the space coordinates.

Now, according to Einstein:

$$R_{\mu\nu} - g_{\mu\nu} R/2 + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}/C^4$$

These equations are known as Einstein Field Equations.

where,

⇒  $R_{\mu\nu}$  is the Ricci curvature tensor

⇒ R is the Ricci scalar

⇒  $g_{\mu\nu}$  is the Metric tensor

⇒ G is the gravitational constant

⇒ C is the speed of Light

⇒  $T_{\mu\nu}$  is the Stress-Momentum-Energy tensor

⇒  $\mu = 0, 1, 2, 3$  and  $\nu = 0, 1, 2, 3$

⇒  $\Lambda$  is the cosmological constant

Einstein's field equations give the relation between space-time and Mass and Energy. In the Einstein equation, LHS defines the Curvature of space-time and RHS defines the Mass and Energy.

When there is no matter present in the space-time matrix, then the energy-momentum tensor vanishes.

Hence, Einstein's equations for the vacuum will be,

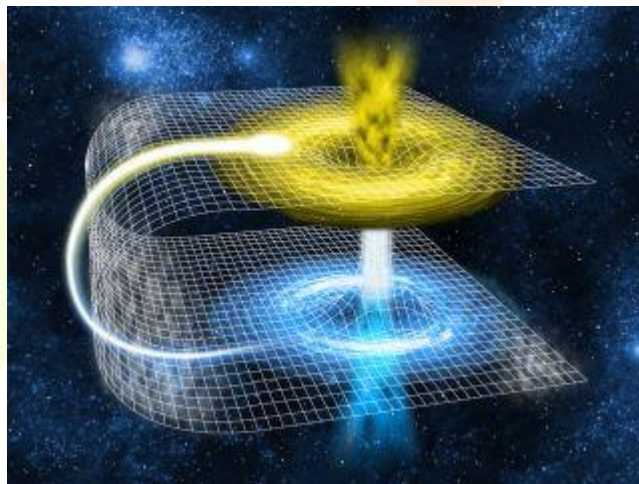
$$\mathbf{R}_{\mu\nu} = \mathbf{g}_{\mu\nu} \mathbf{R}/2 + \Lambda \mathbf{g}_{\mu\nu}$$

Now, for weak fields,  $\Lambda$  (cosmological constant can be neglected as it has a very small value). Hence, we get,

$$\mathbf{R}_{\mu\nu} = \mathbf{g}_{\mu\nu} \mathbf{R}/2 \quad \text{or} \quad \mathbf{R}_{\mu\nu} = 0$$

### Wormhole

A wormhole is a structure linking disparate points in spacetime and is based on the special solution of the Einstein field equations. A wormhole can be visualized as a tunnel with two ends at separate points in space-time (i.e., different locations, different points in time, or both). Wormholes are consistent with the general theory of relativity, but whether wormholes exist remains to be seen.



(Image credit: <http://www.shutterstock.com/gallery-138187p1.html>)>edobric</a> | <http://www.shutterstock.com/>>Shutterstock</a>)

:=) Einstein-Rosen Bridge :

Einstein and Rosen phrased the concept of a bridge that can act as a link between the two points located far in the universe. They suggested that by curving the space-time matrix, this bridge can be made.

Einstein and Rosen discussed two types of bridges:

1). Neutral Bridges: Neutral Einstein-Rosen bridge is the uncharged bridge given by the Schwarzschild co-ordinates,

$$ds^2 = -(1 - 2M/r)dt^2 + dr^2/(1-2M/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

However, this bridge doesn't work for  $M < 0$ .

2). Quasi-charged Bridges: It is an electrically charged bridge of charge  $Q$  and mass  $M$ . In Schwarzschild coordinates,

$$ds^2 = -(1-2M/r+Q^2/r^2)dt^2 + dr^2/(1-2M/r+Q^2/r^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

:=) Morris Thorne Wormhole: Morris and Thorne assumed a traversable wormhole that is spherically symmetrical, time-independent, and non-rotating.

According to Morris and Thorne, Schwarzschild coordinates will be given by:

$$ds^2 = -e^{2\Phi}dt^2 + dr^2/(1-b/r) + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

Where ' $\Phi$ ' and ' $b$ ' are the functions of the radial component ' $r$ '.

->  $\Phi(r)$  represents the Redshift function. It gives information about the change in frequency of electromagnetic radiation in the gravitational field.

->  $b(r)$  represents the shape function. It gives information about the shape and size of the Traversable wormhole.

Now, here given Metric will be taken as spherically symmetrical and static and hence it will be independent of time.

## Calculations for Ricci curvature tensor

We know that the Riemann curvature tensor ( $R^{\alpha}_{\beta\gamma\delta}$ ) is given by,

$$R^{\alpha}_{\beta\gamma\delta} = \partial\Gamma^{\alpha}_{\beta\delta}/\partial x^{\gamma} - \partial\Gamma^{\alpha}_{\beta\gamma}/\partial x^{\delta} + \Gamma^{\alpha}_{\sigma\gamma} \Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta} \Gamma^{\sigma}_{\beta\gamma} \quad \text{----- (1)}$$

$$\text{Where, } \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha m} (\partial g_{m\beta}/\partial x^{\gamma} + \partial g_{m\gamma}/\partial x^{\beta} - \partial g_{\beta\gamma}/\partial x^m) \quad \text{----- (2)}$$

{where, values of  $\alpha, \beta, \gamma, \delta, \sigma$  can be 0, 1, 2, 3 such that  $x^0 = t, x^1 = r, x^2 = \theta$  and  $x^3 = \phi$ }

Now, the Riemann tensor ( $R^{\alpha}_{\beta\gamma\delta}$ ) becomes Ricci Tensor ( $R_{\mu\nu}$ ) if two of its indices become equal i.e  $\alpha = \gamma$ .

Let, Ricci Tensors for the traversable wormhole be:

$R_{00}, R_{11}, R_{22}, R_{33}$  such that,

$$\text{(i) } R_{00} = R^0_{000} + R^1_{010} + R^2_{020} + R^3_{030} \quad \text{----- (3)}$$

$$\text{(ii) } R_{11} = R^0_{101} + R^1_{111} + R^2_{121} + R^3_{131} \quad \text{----- (4)}$$

$$\text{(iii) } R_{22} = R^0_{202} + R^1_{212} + R^2_{222} + R^3_{232} \quad \text{----- (5)}$$

$$\text{(iv) } R_{33} = R^0_{303} + R^1_{313} + R^2_{323} + R^3_{333} \quad \text{----- (6)}$$

Now, we will calculate the values of each Ricci Tensor.

**(a) For  $R_{00}$ :**

$$1) \text{ Taking, } R^0_{000} = \partial\Gamma^0_{00}/\partial x^0 - \partial\Gamma^0_{00}/\partial x^0 + \Gamma^0_{\sigma 0} \Gamma^{\sigma}_{00} - \Gamma^0_{\sigma 0} \Gamma^{\sigma}_{00}$$

Now,

$$\Rightarrow \text{For } \partial\Gamma^0_{00}/\partial x^0,$$

$$\Rightarrow \Gamma^0_{00} = \frac{1}{2} g^{0m} (\partial g_{m0}/\partial x^0 + \partial g_{m0}/\partial x^0 - \partial g_{00}/\partial x^m)$$

$$\Rightarrow \Gamma^0_{00} = 0$$

$$\Rightarrow \partial\Gamma^0_{00}/\partial x^0 = 0 \quad \text{----- (7)}$$

$$\Rightarrow \text{For } \Gamma^0_{\sigma 0} \Gamma^{\sigma}_{00},$$

$$\Gamma^0_{\sigma 0} = 0 = \Gamma^{\sigma}_{00}$$

$$\Rightarrow \Gamma^0_{\sigma 0} \Gamma^{\sigma}_{00} = 0 \quad \text{----- (8)}$$

$$\Rightarrow \text{From (7) and (8), we get, } R^0_{000} = 0 \quad \text{----- (9)}$$

$$2) \text{ Taking, } R^1_{010} = \partial\Gamma^1_{00}/\partial x^1 - \partial\Gamma^1_{01}/\partial x^0 + \Gamma^1_{\sigma 1} \Gamma^{\sigma}_{00} - \Gamma^1_{\sigma 0} \Gamma^{\sigma}_{01}$$



$\Rightarrow$  For  $\partial\Gamma^1_{00}/\partial x^1$ ,

$$\Gamma^1_{00} = \frac{1}{2} g^{1m} (\partial g_{m0}/\partial x^0 + \partial g_{m0}/\partial x^0 - \partial g_{00}/\partial x^m) \text{ ----- (10)}$$

Here, putting  $m=1$ ,

$$\Gamma^1_{00} = \frac{1}{2} g^{11} (-\partial g_{00}/\partial x^1)$$

$$= \frac{1}{2} (1-b/r) [-\partial(-e^{2\Phi})/\partial x^1] \quad (b^I \text{ \& } \Phi^I \text{ are derivatives } b \text{ \& } \Phi)$$

$$\Gamma^1_{00} = \frac{1}{2} (1-b/r)(2\Phi^I)(e^{2\Phi}) \quad (b^{II} \text{ \& } \Phi^{II} \text{ are double derivatives of } b \text{ \& } \Phi)$$

$$\Rightarrow \partial\Gamma^1_{00}/\partial x^1 = (1-b/r)[\Phi^{II}e^{2\Phi} + (\Phi^I)^2 e^{2\Phi}] + [(b-b^I r)/r] (\Phi^I e^{2\Phi})$$

For  $\partial\Gamma^1_{01}/\partial x^0$ ,

$$\partial\Gamma^1_{01}/\partial x^0 = 0 \text{ ----- (11)} \quad (\text{as metric is static})$$

For  $\Gamma^1_{\sigma 1} \Gamma^{\sigma}_{00}$ ,

$$\Gamma^1_{\sigma 1} = \frac{1}{2} g^{1m} (\partial g_{m\sigma}/\partial x^1 + \partial g_{m1}/\partial x^{\sigma} - \partial g_{\sigma 1}/\partial x^m)$$

For  $m=1$  and  $\sigma=1$

$$\Gamma^1_{\sigma 1} = \frac{1}{2} g^{11} (\partial g_{11}/\partial x^1 + \partial g_{11}/\partial x^1 - \partial g_{11}/\partial x^1)$$

$$\Rightarrow \Gamma^1_{\sigma 1} = \frac{1}{2} g^{11} (\partial g_{11}/\partial x^1)$$

$$\Rightarrow \Gamma^1_{\sigma 1} = (1/2r) (rb^I - b)/(r-b)$$

Now,

$$\Gamma^{\sigma}_{00} = \frac{1}{2} g^{\sigma m} (\partial g_{m0}/\partial x^0 + \partial g_{m0}/\partial x^0 - \partial g_{00}/\partial x^m)$$

$$\Rightarrow \Gamma^{\sigma}_{00} = \frac{1}{2} g^{\sigma m} (-\partial g_{00}/\partial x^m)$$

$\Rightarrow$  For  $m=1$  and  $\sigma=1$

$$\Rightarrow \Gamma^{\sigma}_{00} = \frac{1}{2} g^{11} (-\partial g_{00}/\partial x^1)$$

$$\Rightarrow \Gamma^{\sigma}_{00} = (1-b/r) (\Phi^I e^{2\Phi})$$

$$\text{Hence, } \Gamma^1_{\sigma 1} \Gamma^{\sigma}_{00} = 1/2r^2 (rb^I - b) (\Phi^I e^{2\Phi}) \text{ ----- (12)}$$

For  $\Gamma^1_{\sigma 0} \Gamma^{\sigma}_{01}$

$$\Gamma^1_{\sigma 0} = \frac{1}{2} g^{1m} (\partial g_{m\sigma}/\partial x^0 + \partial g_{m0}/\partial x^{\sigma} - \partial g_{00}/\partial x^m)$$

For  $m=1$  and  $\sigma=0$

$$\Gamma^1_{\sigma 0} = \frac{1}{2} g^{11} (\partial g_{10}/\partial x^0 + \partial g_{10}/\partial x^0 - \partial g_{00}/\partial x^1)$$

$$\Gamma^1_{\sigma 0} = \frac{1}{2} g^{11} (-\partial g_{00}/\partial x^1)$$

$$\Gamma^1_{\sigma 0} = (1-b/r) (\Phi^I e^{2\Phi})$$

For  $\Gamma^{\sigma}_{01}$

$$\Gamma^{\sigma}_{01} = \frac{1}{2} g^{\sigma m} (\partial g_{m0}/\partial x^1 + \partial g_{m1}/\partial x^0 - \partial g_{01}/\partial x^m)$$

For  $m = 0$  and  $\sigma = 0$

$$\Gamma^{\sigma}_{01} = \frac{1}{2} g^{00} (\partial g_{00}/\partial x^1 + \partial g_{01}/\partial x^0 - \partial g_{01}/\partial x^0)$$

$$\Gamma^{\sigma}_{01} = \frac{1}{2} g^{00} (\partial g_{00}/\partial x^1)$$

$$\Rightarrow \Gamma^{\sigma}_{01} = \Phi^I$$

Hence, we get

$$\Gamma^1_{\sigma 0} \Gamma^{\sigma}_{01} = (1-b/r)(e^{2\Phi})(\Phi^I)^2 \text{ ----- (13)}$$

From (10), (11), (12), and (13), we get

$$R^1_{010} = e^{2\Phi} [(1-b/r) (\Phi^{II} + (\Phi^I)^2) + (b - b^I r) \Phi^I / 2r^2] \text{ ----- (14)}$$

**3) Now, taking  $R^2_{020} = \partial \Gamma^2_{00}/\partial x^2 - \partial \Gamma^0_{02}/\partial x^0 + \Gamma^2_{\sigma 2} \Gamma^{\sigma}_{00} - \Gamma^2_{\sigma 0} \Gamma^{\sigma}_{02}$**

Now, here  $\partial \Gamma^2_{00}/\partial x^2 = 0$ ,  $\partial \Gamma^0_{02}/\partial x^0 = 0$ , and  $\Gamma^2_{\sigma 0} \Gamma^{\sigma}_{02} = 0$

So, for  $\Gamma^2_{\sigma 2} \Gamma^{\sigma}_{00}$

$$\Rightarrow \Gamma^2_{\sigma 2} = \frac{1}{2} g^{2m} (\partial g_{m\sigma}/\partial x^2 + \partial g_{m2}/\partial x^{\sigma} - \partial g_{\sigma 2}/\partial x^m)$$

For  $m = 2$

$$\Gamma^2_{\sigma 2} = \frac{1}{2} g^{22} (\partial g_{2\sigma}/\partial x^2 + \partial g_{22}/\partial x^{\sigma} - \partial g_{\sigma 2}/\partial x^2)$$

$$\Gamma^2_{\sigma 2} = \frac{1}{2} g^{2m} (\partial g_{22}/\partial x^{\sigma})$$

$$\Rightarrow \Gamma^2_{\sigma 2} = 1/r$$

Now for  $\Gamma^{\sigma}_{00}$

$$\Gamma^{\sigma}_{00} = (1-b/r) (\Phi^I e^{2\Phi})$$

$$\Rightarrow \Gamma^2_{\sigma 2} \Gamma^{\sigma}_{00} = (1-b/r) (\Phi^I e^{2\Phi})/r$$

Hence,  $R^2_{020} = (1-b/r) (\Phi^I e^{2\Phi})/r \text{ ----- (15)}$

**(4) Now, taking  $R^3_{030} = \partial \Gamma^3_{00}/\partial x^3 - \partial \Gamma^3_{03}/\partial x^0 + \Gamma^3_{\sigma 3} \Gamma^{\sigma}_{00} - \Gamma^3_{\sigma 0} \Gamma^{\sigma}_{03}$**

Now, here  $\partial\Gamma^3_{00}/\partial x^3 = 0$  and  $\partial\Gamma^3_{03}/\partial x^0 = 0$  and also  $\Gamma^3_{\sigma 0} \Gamma^\sigma_{03} = 0$

For  $\Gamma^3_{\sigma 3} \Gamma^\sigma_{00}$ ,

$$\Rightarrow \Gamma^\sigma_{00} = (1-b/r) (\Phi^I e^{2\Phi})$$

And,

$$\Rightarrow \Gamma^3_{\sigma 3} = \frac{1}{2} g^{3m} (\partial g_{m\sigma}/\partial x^3 + \partial g_{m3}/\partial x^\sigma - \partial g_{\sigma 3}/\partial x^m)$$

For  $m=3$  and  $\sigma = 1$

$$\Gamma^3_{\sigma 3} = 1/r$$

$$\Rightarrow \Gamma^3_{\sigma 3} \Gamma^\sigma_{00} = (1-b/r) (\Phi^I e^{2\Phi})/r$$

$$\text{Hence, } R^3_{030} = (1-b/r) (\Phi^I e^{2\Phi})/r \text{ ----- (16)}$$

Hence, from (9), (14), (15), and (16) we get,

$$\mathbf{R_{00} = (e^{2\Phi}) [(1-b/r) (\Phi'' + (\Phi')^2) + (b - b'r) \Phi'/2r^2 + 2(1-b/r) (\Phi')/r ]}$$

**(b) For  $R_{11}$ :**

$$(1) \text{ Taking, } R^0_{101} = \partial\Gamma^0_{11}/\partial x^0 - \partial\Gamma^0_{10}/\partial x^1 + \Gamma^0_{\sigma 0} \Gamma^\sigma_{11} - \Gamma^0_{\sigma 1} \Gamma^\sigma_{10}$$

$$\text{Here, } \partial\Gamma^0_{11}/\partial x^0 = 0 \quad \{\text{metric is static}\}$$

$$\text{Now, taking } \partial\Gamma^0_{10}/\partial x^1$$

$$\Rightarrow \Gamma^0_{10} = \frac{1}{2} g^{0m} (\partial g_{m1}/\partial x^0 + \partial g_{m0}/\partial x^1 - \partial g_{10}/\partial x^m)$$

For  $m = 0$ ,

$$\Gamma^0_{10} = \frac{1}{2} g^{00} (\partial g_{01}/\partial x^0 + \partial g_{00}/\partial x^1 - \partial g_{10}/\partial x^0)$$

$$\Rightarrow \Gamma^0_{10} = \frac{1}{2} g^{00} (\partial g_{00}/\partial x^1)$$

$$\Rightarrow \Gamma^0_{10} = \Phi^I$$

$$\Rightarrow \partial\Gamma^0_{10}/\partial x^1 = \Phi^{II} \text{ ----- (17)}$$

Now, taking  $\Gamma^0_{\sigma 0} \Gamma^\sigma_{11}$

$$\Gamma^0_{\sigma 0} = \frac{1}{2} g^{0m} (\partial g_{m\sigma}/\partial x^0 + \partial g_{m0}/\partial x^\sigma - \partial g_{\sigma 0}/\partial x^m)$$

For  $m = 0$ ,

$$\Gamma^0_{\sigma 0} = \frac{1}{2} g^{00} (\partial g_{0\sigma}/\partial x^0 + \partial g_{00}/\partial x^\sigma - \partial g_{\sigma 0}/\partial x^0)$$



For  $\sigma = 1$ ,

$$\Gamma^0_{\sigma 0} = \frac{1}{2} g^{00} (\partial g_{00} / \partial x^1)$$

$$\Gamma^0_{\sigma 0} = \Phi^I$$

And,

$$\Gamma^{\sigma}_{11} = \frac{1}{2} g^{\sigma m} (\partial g_{m1} / \partial x^1 + \partial g_{m1} / \partial x^1 - \partial g_{11} / \partial x^m)$$

For  $m = 1$ ,

$$\Gamma^{\sigma}_{11} = \frac{1}{2} g^{\sigma 1} (\partial g_{11} / \partial x^1 + \partial g_{11} / \partial x^1 - \partial g_{11} / \partial x^1)$$

For  $\sigma = 1$

$$\Gamma^{\sigma}_{11} = \frac{1}{2} g^{11} (\partial g_{11} / \partial x^1)$$

$$\Rightarrow \Gamma^{\sigma}_{11} = (b^I r - b) / 2r(r-b)$$

$$\Rightarrow \Gamma^0_{\sigma 0} \Gamma^{\sigma}_{11} = \Phi^I (b^I r - b) / 2r(r-b) \text{ ----- (18)}$$

Now, taking  $\Gamma^0_{\sigma 1} \Gamma^{\sigma}_{10}$

$$\Gamma^0_{\sigma 1} = \frac{1}{2} g^{0m} (\partial g_{m\sigma} / \partial x^1 + \partial g_{m1} / \partial x^{\sigma} - \partial g_{\sigma 1} / \partial x^m)$$

For  $m = 0$  and  $\sigma = 0$ ,

$$\Gamma^0_{\sigma 1} = \frac{1}{2} g^{00} (\partial g_{00} / \partial x^1)$$

$$\Gamma^0_{\sigma 1} = \Phi^I$$

Now,

$$\Gamma^{\sigma}_{10} = \frac{1}{2} g^{\sigma m} (\partial g_{m1} / \partial x^0 + \partial g_{m0} / \partial x^1 - \partial g_{10} / \partial x^m)$$

For  $m = 0$  and  $\sigma = 0$

$$\Gamma^{\sigma}_{10} = \frac{1}{2} g^{00} (\partial g_{00} / \partial x^1)$$

$$\Gamma^{\sigma}_{10} = \Phi^I$$

$$\Rightarrow \Gamma^0_{\sigma 1} \Gamma^{\sigma}_{10} = (\Phi^I)^2 \text{ ----- (19)}$$

From (17), (18), and (19), we get,

$$R^0_{101} = -\Phi^{II} - (\Phi^I)^2 + \Phi^I (b^I r - b) / 2r(r-b) \text{ ----- (20)}$$

(2) Taking,  $R^1_{111} = \partial \Gamma^1_{11} / \partial x^1 - \partial \Gamma^1_{11} / \partial x^1 + \Gamma^1_{\sigma 1} \Gamma^{\sigma}_{11} - \Gamma^1_{\sigma 1} \Gamma^{\sigma}_{11}$

$$\Rightarrow R^1_{111} = 0 \text{ ----- (21)}$$

**(3)** Taking,  $R^2_{121} = \partial\Gamma^2_{11}/\partial x^2 - \partial\Gamma^2_{12}/\partial x^1 + \Gamma^2_{\sigma 2} \Gamma^\sigma_{11} - \Gamma^2_{\sigma 1} \Gamma^\sigma_{12}$

For  $\partial\Gamma^2_{11}/\partial x^2$ ,

$$\Gamma^2_{11} = 0 \Rightarrow \partial\Gamma^2_{11}/\partial x^2 = 0 \text{ ----- (22)}$$

For  $\partial\Gamma^2_{12}/\partial x^1$ ,

$$\Rightarrow \Gamma^2_{12} = \frac{1}{2} g^{2m} (\partial g_{m1}/\partial x^2 + \partial g_{m2}/\partial x^1 - \partial g_{12}/\partial x^m)$$

For  $m = 2$ ,

$$\Gamma^2_{12} = \frac{1}{2} g^{22} (\partial g_{21}/\partial x^2 + \partial g_{22}/\partial x^1 - \partial g_{12}/\partial x^2)$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} (\partial g_{22}/\partial x^1)$$

$$\Rightarrow \Gamma^2_{12} = 1/r$$

$$\Rightarrow \partial\Gamma^2_{12}/\partial x^1 = -1/r^2 \text{ ----- (23)}$$

Now, for  $\Gamma^2_{\sigma 2} \Gamma^\sigma_{11}$

$$\Rightarrow \Gamma^2_{\sigma 2} = \frac{1}{2} g^{2m} (\partial g_{m\sigma}/\partial x^2 + \partial g_{m2}/\partial x^\sigma - \partial g_{\sigma 2}/\partial x^m)$$

For  $m = 2$  and  $\sigma = 1$ ,

$$\Gamma^2_{\sigma 2} = \frac{1}{2} g^{22} (\partial g_{21}/\partial x^2 + \partial g_{22}/\partial x^1 - \partial g_{12}/\partial x^2)$$

$$\Gamma^2_{\sigma 2} = \frac{1}{2} g^{22} (\partial g_{22}/\partial x^1)$$

$$\Gamma^2_{\sigma 2} = 1/r$$

$$\Rightarrow \Gamma^\sigma_{11} = \frac{1}{2} g^{\sigma m} (\partial g_{m1}/\partial x^1 + \partial g_{m1}/\partial x^1 - \partial g_{11}/\partial x^m)$$

For  $m = 1 = \sigma$ ,

$$\Gamma^\sigma_{11} = \frac{1}{2} g^{11} (\partial g_{11}/\partial x^1 + \partial g_{11}/\partial x^1 - \partial g_{11}/\partial x^1)$$

$$\Gamma^\sigma_{11} = \frac{1}{2} g^{11} (\partial g_{11}/\partial x^1)$$

$$\Rightarrow \Gamma^\sigma_{11} = \frac{1}{2} (b^1 r - b)/r (r - b)$$

$$\Rightarrow \Gamma^2_{\sigma 2} \Gamma^\sigma_{11} = \frac{1}{2} (b^1 r - b)/r^2 (r - b) \text{ ----- (24)}$$

Now, For,  $\Gamma^2_{\sigma 1} \Gamma^\sigma_{12}$

$$\Rightarrow \Gamma^2_{\sigma 1} = \frac{1}{2} g^{2m} (\partial g_{m\sigma}/\partial x^1 + \partial g_{m1}/\partial x^\sigma - \partial g_{\sigma 1}/\partial x^m)$$

For  $m = 2$ ,

$$\Gamma^2_{\sigma 1} = \frac{1}{2} g^{22} (\partial g_{2\sigma} / \partial x^1 + \partial g_{21} / \partial x^\sigma - \partial g_{\sigma 1} / \partial x^2)$$

For  $\sigma = 2$ ,

$$\Gamma^2_{\sigma 1} = \frac{1}{2} g^{22} (\partial g_{22} / \partial x^1 + \partial g_{21} / \partial x^2 - \partial g_{\sigma 1} / \partial x^2)$$

$$\Gamma^2_{\sigma 1} = \frac{1}{2} g^{22} (\partial g_{22} / \partial x^1)$$

$$\Gamma^2_{\sigma 1} = 1/r$$

Now,

$$\Gamma^\sigma_{12} = \frac{1}{2} g^{\sigma m} (\partial g_{m1} / \partial x^2 + \partial g_{m2} / \partial x^1 - \partial g_{12} / \partial x^m)$$

For  $m=2=\sigma$ ,

$$\Gamma^\sigma_{12} = \frac{1}{2} g^{22} (\partial g_{21} / \partial x^2 + \partial g_{22} / \partial x^1 - \partial g_{12} / \partial x^2) = \frac{1}{2} g^{22} (\partial g_{22} / \partial x^1)$$

$$\Gamma^\sigma_{12} = 1/r$$

$$\Rightarrow \Gamma^2_{\sigma 1} \Gamma^\sigma_{12} = 1/r^2 \text{ ----- (25)}$$

Hence, from (22), (23), (24), and (25), we get,

$$R^2_{121} = \frac{1}{2} (b^1 r - b)/r (r - b) \text{ ----- (26)}$$

**(4) Taking,  $R^3_{131} = \partial \Gamma^3_{11} / \partial x^3 - \partial \Gamma^3_{13} / \partial x^1 + \Gamma^3_{\sigma 3} \Gamma^\sigma_{11} - \Gamma^3_{\sigma 1} \Gamma^\sigma_{13}$**

Now, for  $\Gamma^3_{11} = 0 \Rightarrow \partial \Gamma^3_{11} / \partial x^3 = 0 \text{ ----- (27)}$

For  $\partial \Gamma^3_{13} / \partial x^1$ ,

$$\Gamma^3_{13} = \frac{1}{2} g^{3m} (\partial g_{m1} / \partial x^3 + \partial g_{m3} / \partial x^1 - \partial g_{13} / \partial x^m)$$

For  $m = 3$ ,

$$\Gamma^3_{13} = 1/r \Rightarrow \partial \Gamma^3_{13} / \partial x^1 = -1/r^2 \text{ ----- (28)}$$

Now for  $\Gamma^3_{\sigma 3} \Gamma^\sigma_{11}$

$$\Rightarrow \Gamma^3_{\sigma 3} = \frac{1}{2} g^{3m} (\partial g_{m\sigma} / \partial x^3 + \partial g_{m3} / \partial x^\sigma - \partial g_{\sigma 3} / \partial x^m)$$

For  $m=3$  and  $\sigma = 1$ ,

$$\Rightarrow \Gamma^3_{\sigma 3} = \frac{1}{2} g^{33} (\partial g_{33} / \partial x^1)$$

$$\Rightarrow \Gamma^3_{\sigma 3} = 1/r$$

$$\Gamma^{\sigma}_{11} = \frac{1}{2} g^{\sigma m} (\partial g_{m1}/\partial x^1 + \partial g_{m1}/\partial x^1 - \partial g_{11}/\partial x^m)$$

$$\text{For } m = 1 = \sigma, \Rightarrow \Gamma^{\sigma}_{11} = \frac{1}{2} g^{11} (\partial g_{11}/\partial x^1)$$

$$\Rightarrow \Gamma^3_{\sigma 3} \Gamma^{\sigma}_{11} = \frac{1}{2} (b^1 r - b)/r^2 (r - b) \text{ ----- (29)}$$

$$\text{For } \Gamma^3_{\sigma 1} \Gamma^{\sigma}_{13},$$

$$\Gamma^3_{\sigma 1} = \frac{1}{2} g^{3m} (\partial g_{m\sigma}/\partial x^1 + \partial g_{m1}/\partial x^{\sigma} - \partial g_{\sigma 1}/\partial x^m)$$

$$\text{For } m = 3 = \sigma$$

$$\Gamma^3_{\sigma 1} = \frac{1}{2} g^{33} (\partial g_{33}/\partial x^1 + \partial g_{31}/\partial x^3 - \partial g_{31}/\partial x^3) = \frac{1}{2} g^{33} (\partial g_{33}/\partial x^1)$$

$$\Gamma^3_{\sigma 1} = 1/r$$

Now,

$$\Gamma^{\sigma}_{13} = \frac{1}{2} g^{\sigma m} (\partial g_{m1}/\partial x^3 + \partial g_{m3}/\partial x^1 - \partial g_{13}/\partial x^m)$$

$$\text{For } m = 3 = \sigma$$

$$\Gamma^{\sigma}_{13} = \frac{1}{2} g^{33} (\partial g_{31}/\partial x^3 + \partial g_{33}/\partial x^1 - \partial g_{13}/\partial x^3) = \frac{1}{2} g^{33} (\partial g_{33}/\partial x^1)$$

$$\Gamma^{\sigma}_{13} = 1/r$$

$$\Rightarrow \Gamma^3_{\sigma 1} \Gamma^{\sigma}_{13} = 1/r^2 \text{ ----- (30)}$$

From (27), (28), (29), and (30), we get,

$$R^3_{131} = \frac{1}{2} (b^1 r - b)/r^2 (r - b) \text{ ----- (31)}$$

From, (20), (21), (26), and (30), we get,

$$R_{11} = (1-b/r)^{-1} [(1-b/r)(-\Phi^{11} - (\Phi^1)^2) + \Phi^1 (b^1 r - b)/2r^2 + (b^1 r - b)/r^3]$$

**(c) For  $R_{22}$ :**

$$(1) \text{ Taking, } R^0_{202} = \partial \Gamma^0_{22}/\partial x^0 - \partial \Gamma^0_{20}/\partial x^2 + \Gamma^0_{\sigma 0} \Gamma^{\sigma}_{22} - \Gamma^0_{\sigma 2} \Gamma^{\sigma}_{20}$$

$$\text{Here, } \partial \Gamma^0_{22}/\partial x^0 = 0 \text{ ----- (32)}$$

$$\text{Also, } \Gamma^0_{\sigma 2} = 0 \Rightarrow \Gamma^0_{\sigma 2} \Gamma^{\sigma}_{20} = 0 \text{ ----- (33)}$$

$$\text{Also, } \Gamma^0_{20} = 0 \Rightarrow \partial \Gamma^0_{20}/\partial x^2 = 0 \text{ ----- (34)}$$

$$\text{For } \Gamma^0_{\sigma 0} \Gamma^{\sigma}_{22}$$

$$\Rightarrow \Gamma^0_{\sigma 0} = \frac{1}{2} g^{0m} (\partial g_{m\sigma} / \partial x^0 + \partial g_{m0} / \partial x^\sigma - \partial g_{\sigma 0} / \partial x^m)$$

By putting  $m=0$  and  $\sigma = 1$ , we get,

$$\Gamma^0_{\sigma 0} = \Phi^I$$

And,

$$\Gamma^\sigma_{22} = \frac{1}{2} g^{\sigma m} (\partial g_{m2} / \partial x^2 + \partial g_{m2} / \partial x^2 - \partial g_{22} / \partial x^m)$$

Putting  $m=1$  and  $\sigma = 1$ , we get,

$$\Gamma^\sigma_{22} = -r(1-b/r)$$

$$\Rightarrow \Gamma^0_{\sigma 0} \Gamma^\sigma_{22} = -r \Phi^I (1-b/r) \text{ ----- (35)}$$

From, (32), (33), (34), and (35), we get,

$$R^0_{202} = -r \Phi^I (1-b/r) = r^2 [-\Phi^I (1-b/r)/r] \text{ ----- (36)}$$

$$(2) \text{ Taking, } R^1_{212} = \partial \Gamma^1_{22} / \partial x^1 - \partial \Gamma^1_{21} / \partial x^2 + \Gamma^1_{\sigma 1} \Gamma^\sigma_{22} - \Gamma^1_{\sigma 2} \Gamma^\sigma_{21}$$

$$\text{Now here, } \partial \Gamma^1_{21} / \partial x^2 = 0 \text{ ----- (37)}$$

Here, for  $\partial \Gamma^1_{22} / \partial x^1$ ,

$$\Gamma^1_{22} = b - r \Rightarrow \partial \Gamma^1_{22} / \partial x^1 = b^I - 1 \text{ ----- (38)}$$

Now for  $\Gamma^1_{\sigma 1} \Gamma^\sigma_{22}$ ,

$$\Gamma^1_{\sigma 1} = \frac{1}{2} (b^I r - b) / (r - b)$$

And,

$$\Gamma^\sigma_{22} = b - r$$

$$\Rightarrow \Gamma^1_{\sigma 1} \Gamma^\sigma_{22} = \frac{1}{2} (b^I r - b) / r \text{ ----- (39)}$$

For  $\Gamma^1_{\sigma 2} \Gamma^\sigma_{21}$ , we have

$$\Gamma^1_{\sigma 2} = b - r$$

And,

$$\Gamma^\sigma_{21} = 1/r$$

$$\Rightarrow \Gamma^1_{\sigma 2} \Gamma^\sigma_{21} = (b - r) / r \text{ ----- (40)}$$

From (37), (38), (39), and (40), we get,

$$R^1_{212} = (b^I r - b) / 2r \text{ ----- (41)}$$

$$(3) \text{ Taking, } R^2_{222} = \partial \Gamma^2_{22} / \partial x^2 - \partial \Gamma^2_{22} / \partial x^2 + \Gamma^2_{\sigma 2} \Gamma^\sigma_{22} - \Gamma^2_{\sigma 2} \Gamma^\sigma_{22}$$



$$\Rightarrow R^2_{222} = 0 \text{ ----- (42)}$$

$$(4) \text{ Taking, } R^3_{232} = \partial \Gamma^3_{22} / \partial x^3 - \partial \Gamma^3_{23} / \partial x^2 + \Gamma^3_{\sigma 3} \Gamma^{\sigma}_{22} - \Gamma^3_{\sigma 2} \Gamma^{\sigma}_{23}$$

$$\text{We get, } R^3_{232} = b/r \text{ ----- (43)}$$

From (36), (41), (42), and (43), we get,

$$\mathbf{R}_{22} = r^2 [-\Phi^I (1-b/r)/r + \frac{1}{2} (b^I r - b)/r^3 + b/r^3]$$

Similarly, we have

$$\mathbf{R}_{33} = \sin^2 \theta \mathbf{R}_{22}$$

### Calculations for Ricci scalar

Here, R will be the Ricci scalar,

we know,

$$\mathbf{R}_{\mu\nu} = g_{\mu\nu} \mathbf{R}/2$$

$$\Rightarrow \mathbf{R} = 2g^{\mu\nu} \mathbf{R}_{\mu\nu}$$

$$\Rightarrow \mathbf{R} = 2[g^{00} \mathbf{R}_{00} + g^{11} \mathbf{R}_{11} + g^{22} \mathbf{R}_{22} + g^{33} \mathbf{R}_{33}]$$

$$\begin{aligned} \Rightarrow \mathbf{R} = & -[(1-b/r) (\Phi^{II} + (\Phi^I)^2) + (b - b^I r) \Phi^I / 2r^2 + 2(1-b/r) (\Phi^I)/r] + \\ & [(1-b/r)(-\Phi^{II} - (\Phi^I)^2) + \Phi^I (b^I r - b) / 2r^2 + (b^I r - b) / r^3] + \\ & [-\Phi^I (1-b/r)/r + \frac{1}{2} (b^I r - b)/r^3 + b/r^3] + [-\Phi^I (1-b/r)/r + \\ & \frac{1}{2} (b^I r - b)/r^3 + b/r^3] \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathbf{R} = & [-4(1-b/r) (\Phi^{II} + (\Phi^I)^2) + 2\Phi^I (b^I r - b) / r^2 - 8(1-b/r) (\Phi^I)/r + \\ & 4(b^I r - b) / r^3 + 4b/r^3] \end{aligned}$$

Hence, we get,

$$\begin{aligned} \mathbf{R} = & -4(1-b/r) (\Phi^{II} + (\Phi^I)^2) + 2\Phi^I (b^I r - b) / r^2 - 8(1-b/r) (\Phi^I)/r + \\ & 4(b^I r - b) / r^3 + 4b/r^3 \end{aligned}$$

## Summary

From calculations, I get the equations of Ricci Curvature tensor for the traversable wormhole and they are as follow:

$$(1) R_{00} = (e^{2\Phi}) [(1-b/r) (\Phi'' + (\Phi')^2) + (b - b'r) \Phi'/2r^2 + 2(1-b/r)(\Phi')/r]$$

$$(2) R_{11} = (1-b/r)^{-1} [(1-b/r)(-\Phi'' - (\Phi')^2) + \Phi' (b'r - b)/2r^2 + (b'r - b)/r^3]$$

$$(3) R_{22} = r^2 [-\Phi' (1-b/r)/r + \frac{1}{2} (b'r - b)/r^3 + b/r^3]$$

$$(4) R_{33} = \sin^2\theta R_{22}$$

Through calculations, I get the following equation of Ricci scalar for the traversable wormhole:

$$R = -4(1-b/r) (\Phi'' + (\Phi')^2) + 2\Phi' (b'r - b)/r^2 - 8(1-b/r) (\Phi')/r + 4(b'r - b)/r^3 + 4b/r^3$$

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